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Contando infinidades - El ”número” de puntos malla en conos convexos
Counting infinities - The ”number” of lattice points on convex cones

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Resumen Este es un tratado informal no dirigido a expertos – basado en una versión más elaborada y extensa cite{GPZ5} así como en trabajos previos de los autores cite{GPZ1,GPZ2,GPZ3,GPZ4} – de como métodos de la física pueden ser empleados para ”contar” un número infinito de puntos. Comenzaremos con el caso clásico del conteo de puntos enteros en el eje real no negativo y la fórmula clásica de Euler-Maclaurin. Después procederemos al conteo de puntos malla en conos producto donde se pueden apreciar, en un arreglo relativamente simple, los roles de la coalgebra y de la factorización algebraica de Birkhoff. Terminaremos con una nota acerca de la generalización a conos convexos.

Abstract This is a very informal account addressed to non-experts – based on a more elaborate and extended version [10] as well as on previous work by the authors [6, 7, 8, 9] – on how methods borrowed from physics can be used to ”count” an infinite number of points. We begin with the classical case of counting integer points on the non-negative real axis and the classical Euler-Maclaurin formula. We then proceed to counting lattice points on product cones where the roles played by the coalgebra and the algebraic Birkhoff factorization can be appreciated in a relatively simple setting. We end the presentation with a note on the generalization to convex cones.

Palabras Clave
cones, coalgebra, renormalization, Birkhoff decomposition, Euler-Maclaurin formula, meromorphic functions

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1. Introduction

Çounting”lattice points on cones which might a priori seem like a very specific issue, actually brings together i) renormalization methods à la Connes and Kreimer [4] borrowed from quantum field theory in the form of algebraic Birkhoff factorization, ii) the Euler-Maclaurin formula on cones and hence on polytopes [2] used to study the geometry of toric varieties, iii) number theory with the conical zeta values (introduced in [7]) that generalize multiple zeta values [13, 17], and which arise in our context as the Taylor coefficients of the interpolator in the Euler-Maclaurin formula. We hope that this very informal presentation which does not claim to be neither exhaustive nor new since it relies on previous work by the authors, will act as an incentive for the lay reader to get further acquainted with renormalization methods. It is based on a more extended version [10].

With the aim in mind of çounting”lattice points on rational polyhedral convex cones, we start from the one dimensional cone $\mathbb{R}_+$ with lattice points given by the positive integers. The ”number.ºf non-negative integers can be derived using an approximation $S(\varepsilon)=\sum_{n=0}^{\infty}e^{-\varepsilon n}$ by an exponential sum. Its analytic extension (denoted by the same symbol S) presents a simple pole at $\varepsilon=0$ with residue 1 so that $S(\varepsilon)=\frac{1}{\varepsilon}+S_+(\varepsilon)$ where S_+ is holomorphic at zero. Coincidentally, the ”polar part”$\frac{1}{\varepsilon}$ equals the integral $I(\varepsilon)=\int_0^{\infty}e^{-\varepsilon x}dx$ leading to the Euler-Maclaurin formula $S=I+\mu$ which relates the sum and the integral of the map $x\mapsto e^{-\varepsilon x}$ by means of the interpolator $\mu=S_+$. Using the terminology borrowed from physicists, we refer to the decomposition $S(\varepsilon)=\frac{1}{\varepsilon}+S_+(\varepsilon)$ into a ”polar part”$\frac{1}{\varepsilon}$ and a holomorphic part $S_+(\varepsilon)$ as the minimal subtraction scheme applied to S. For this particular function, it coincides with the Euler-Maclaurin formula and we have $S_+(0)=\mu(0)=\zeta(0)+1=\frac{1}{2}$.

The coincidence in the case of the discrete exponential sum, between the minimal subtraction scheme and the Euler-Maclaurin formula, carries out to higher dimensions.

We indeed proceed to çounting”the lattice points $\mathbb{Z}^k_{\geq 0}$ of a (closed) product cone $\mathbb{R}^k_{\geq 0}$ of dimension $k\in\mathbb{N}$. One expects the "number."of points of $\mathbb{Z}^k_{\geq 0}$ to be the k-th power of the "number."of points of $\mathbb{Z}_{\geq 0}$ and this is indeed the case provided one çounts carefully". By this we mean that one should not naively evaluate the "holomorphic part."of the k-th power $S^k(\varepsilon)$ at zero of the exponential sum but instead take the k-th power $S^k_+(0)$ of the holomorphic part S_+ evaluated at zero, which is a straightforward procedure in the rather trivial case of product cones.

We briefly discuss a general algebraic construction which derives S^k_+ from S^k, known as the algebraic Birkhoff factorization that can be viewed as a generalization to higher dimensions of the minimal subtraction scheme mentioned above. It relies on a coproduct on (product) cones built from a complement map which separates a face of the cone from the remaining faces. When applied to the multivariable exponential sum $\widetilde{S}_k : (\varepsilon_1, \cdots, \varepsilon_k) \mapsto \prod_{i=1}^k S(\varepsilon_i)$ on the product cone $\mathbb{R}^k_{\geq 0}$, the general algebraic Birkhoff factorization on coalgebras gives $(\varepsilon_1, \cdots, \varepsilon_k) \mapsto \prod_{i=1}^k S_+(\varepsilon_i)$ as the renormalized holomorphic"part of the map $\widetilde{S}_k$. This algebraic Birkhoff factorization can also be interpreted as an Euler-Maclaurin formula for it factorizes the sum as a (convolution) product of integrals and interpolators on product cones, thus generalizing to higher dimensional cones the equivalence observed in one dimension of the Euler Maclaurin formula with the minimal subtraction scheme.

2. Counting integers

We want to count the non-negative integer points i.e. to evaluate the ill-defined sum "$1+1+\cdots+1+\cdots = \sum_{n=0}^{\infty} n^0$."and more generally the no better defined sum $\sum_{n=0}^{\infty} n^k$ for any non-negative integer k.

2.1 Approximated sums over integers

We first approximate these ill-defined sums; there are various ways to do so [1]. Here we focus on the **heat-kernel type regularization** which approximates the summand by an exponential expression. For positive ε we set

$$S(\varepsilon) := \sum_{n=0}^{\infty} e^{-\varepsilon n}. \tag{1}$$

Let $S_k(\varepsilon) := \sum_{n=0}^{\infty} n^k e^{-\varepsilon n} = (-1)^k \partial_k S(\varepsilon)$.

The sum $S(\varepsilon) = \frac{1}{1-e^{-\varepsilon}}$ can be expressed in terms of the **Todd function**

$$\mathrm{Td}(\varepsilon) := \frac{\varepsilon}{e^{\varepsilon}-1} \tag{2}$$

as

$$S(\varepsilon) = \frac{\mathrm{Td}(-\varepsilon)}{\varepsilon}.$$

The **Todd function** is the exponential generating function for the **Bernoulli numbers** that correspond to the Taylor coefficients.

$$\mathrm{Td}(\varepsilon) = \sum_{n=0}^{\infty} B_n \frac{\varepsilon^n}{n!}. \tag{3}$$

[1]We refer the reader to [15] for a more detailed description of these various regularization methods.

We have

$$\mathrm{Td}(\varepsilon) = \frac{\varepsilon}{e^{\varepsilon}-1} = \frac{\varepsilon}{\varepsilon + \frac{\varepsilon^2}{2} + o(\varepsilon^2)} = \frac{1}{1+\frac{\varepsilon}{2}+o(\varepsilon)} = 1 - \frac{\varepsilon}{2} + o(\varepsilon)$$

so $B_0 = 1; B_1 = -\frac{1}{2}$. Since $\frac{\varepsilon}{e^{\varepsilon}-1} + \frac{\varepsilon}{2} = \frac{\varepsilon}{2}\frac{e^{\frac{\varepsilon}{2}}+e^{-\frac{\varepsilon}{2}}}{e^{\frac{\varepsilon}{2}}-e^{-\frac{\varepsilon}{2}}}$ is an even function, $B_{2k+1} = 0$ for any positive integer k.

Consequently, for any positive integer K we have

$$\mathrm{Td}(\varepsilon) = 1 - \frac{\varepsilon}{2} + \sum_{k=1}^{K} \frac{B_{2k}}{(2k)!}\varepsilon^{2k} + o(\varepsilon^{2K}) \tag{4}$$

and

$$S(\varepsilon) = \frac{\mathrm{Td}(-\varepsilon)}{\varepsilon} = \frac{1}{\varepsilon} + \frac{1}{2} + \sum_{k=1}^{K} \frac{B_{2k}}{(2k)!}\varepsilon^{2k-1} + o(\varepsilon^{2K}). \tag{5}$$

2.2 The one-dimensional Euler-Maclaurin formula

As a consequence of formula (5), the discrete sum $S(\varepsilon) := \sum_{k=0}^{\infty} e^{-\varepsilon k} = \frac{1}{1-e^{-\varepsilon}}$ for positive ε relates to the integral

$$I(\varepsilon) := \int_0^{\infty} e^{-\varepsilon x}\, dx = \frac{1}{\varepsilon} \tag{6}$$

by means of the interpolator

$$\mu(\varepsilon) := S(\varepsilon) - I(\varepsilon) = S(\varepsilon) - \frac{1}{\varepsilon} = \frac{1}{2} + \sum_{k=1}^{K} \frac{B_{2k}}{(2k)!}\varepsilon^{2k-1} + o(\varepsilon^{2K}) \quad \text{for all } K \in \mathbb{N},$$

which is holomorphic at $\varepsilon = 0$. This interpolation formula between the sum and the integral

$$S(\varepsilon) = I(\varepsilon) + \mu(\varepsilon) \tag{7}$$

is a special instance of the Euler-Maclaurin formula for which we refer the reader to [12].

2.3 Evaluating meromorphic functions at poles

Let $\mathrm{Mer}_0(\mathbb{C})$ be the set of germs of meromorphic functions with a simple pole at zero [2]. Let $\mathrm{Hol}_0(\mathbb{C})$ be the set of germs of holomorphic functions at zero.

If f in $\mathrm{Mer}_0(\mathbb{C})$ reads $f(z) = \sum_{i=-1}^{\infty} a_i z^i$, we set $\mathrm{Res}_0(f) := a_{-1}$, called the **complex residue** of f at zero.

The projection map

$$\begin{array}{rcl} \pi_+ : \mathrm{Mer}_0(\mathbb{C}) & \to & \mathrm{Hol}_0(\mathbb{C}) \\ f & \mapsto & \left(z \mapsto f(z) - \dfrac{\mathrm{Res}_0(f)}{z}\right) \end{array}$$

[2]i.e. equivalence classes of meromorphic functions defined on a neighborhood of zero for the equivalence relation $f \sim g$ if f and g coincide on some open neighborhood of zero.

for $f \in \mathrm{Mer}_0(\mathbb{C})$ corresponds to what physicists call a **minimal subtraction scheme**. Whereas $\pi_+(f)$ corresponds to the holomorphic part of f, $\pi_-(f) := (1-\pi_+)(f)$ corresponds to the "polar part" of f.
With the notation of the previous paragraph, we have

$$S_+ := \pi_+ \circ S(\varepsilon) = \mu(\varepsilon); \quad S_- := \pi_- \circ S(\varepsilon) = I(\varepsilon). \tag{8}$$

Thus the Euler-Maclaurin formula (7) amounts to the minimal subtraction scheme applied to S:

$$S = S_+ + S_- = \mu + I. \tag{9}$$

Combining the evaluation map at zero $\mathrm{ev}_0 : f \mapsto f(0)$ on holomorphic germs at zero with the map π_+ provides a regularized evaluator at zero on $\mathrm{Mer}_0(\mathbb{C})$. The map

$$\begin{array}{rcl} \mathrm{ev}_0^{\mathrm{reg}} : \mathrm{Mer}_0(\mathbb{C}) & \to & \mathbb{C} \\ f & \mapsto & \mathrm{ev}_0 \circ \pi_+(f), \end{array} \tag{10}$$

is a linear form that extends the ordinary evaluation map ev_0 defined on the space $\mathrm{Hol}_0(\mathbb{C})$.

Example 1 *We have*

$$\mathrm{ev}_0^{\mathrm{reg}}(S) = \mu(0) = \frac{1}{2} = 1 + B_1.$$

This provides one way of evaluating the number of non negative integer points, which we found is equal to $\frac{1}{2}$, a non integer number! Since the number of positive integers is one unit smaller, it is $-\frac{1}{2}$, a negative non integer number!

3. Counting lattice points on product cones

Given a positive integer k, we now want to çount"the number "$\left(\sum_{\vec{n}\in\mathbb{Z}^k_{\geq 0}} \vec{n}^{\vec{0}}\right)$" of lattice points $\vec{n} \in \mathbb{Z}^k_{\geq 0}$ in the product cone $\mathbb{R}^k_{\geq 0}$, where for $\vec{n} = (n_1, \cdots, n_k) \in \mathbb{Z}^k_{\geq 0}$ and $\vec{r} = (r_1, \cdots, r_k) \in \mathbb{Z}^k_{\geq 0}$ we have set $\vec{n}^{\vec{r}} = n_1^{r_1} \cdots n_k^{r_k}$. We first describe the algebra of product cones.

3.1 Product cones

Let $\mathscr{B}_n = (e_1, \cdots, e_n)$ be the canonical basis of $\mathbb{R}^n$ and $\mathscr{P}(\mathbb{R}^n)$ be the set of **product cones**

$$\langle e_I \rangle := \sum_{i \in I} \mathbb{R}_{\geq 0} e_i, \quad I \subseteq [n] := \{1, \cdots, n\},$$

viewed as subsets of $\mathbb{R}^n$. Extending this basis to a basis $\mathscr{B}_{n+1} = (e_1, \cdots, e_{n+1})$ of $\mathbb{R}^{n+1}$, a product cone in $\mathbb{R}^n$ can be viewed as a product cone in $\mathbb{R}^{n+1}$. We define the set of product cones in $\mathbb{R}^\infty$

$$\mathscr{P}(\mathbb{R}^\infty) = \{\langle e_I \rangle \,|\, I \subset \mathbb{N} \text{ finite } \} \text{ with } \langle e_\emptyset \rangle := \{0\}.$$

3.2 Separable meromorphic functions

The linear map $\mathrm{ev}_0^{\mathrm{reg}} : \mathrm{Mer}_0(\mathbb{C}) \to \mathbb{C}$ extends multiplicatively to the subspace $\mathrm{Mer}_{\mathrm{sep}}(\mathbb{C}^\infty)$ of $\mathrm{Mer}_0(\mathbb{C}^\infty)$ spanned by separable functions [3]:

$$\left\{ f = \prod_{i \in I} f_i \,\middle|\, I \subseteq \mathbb{N} \,\text{finite}, \, f_i \in \mathrm{Mer}_0(\mathbb{C} e_i) \right\}$$

by

$$\mathrm{ev}_{\vec{0}}^{\mathrm{ren}} \left(\prod_{i \in I} f_i \right) := \prod_{i \in I} \mathrm{ev}_0^{\mathrm{reg}}(f_i). \tag{11}$$

3.3 The exponential summation and integration map on product cones

The summation map (1) and the integration map (6), which lie in the linear space $\mathrm{Mer}_0(\mathbb{C})$ of meromorphic germs in one complex variable with a simple pole at zero, induce linear maps on the linear space $\mathbb{R}\mathscr{P}(\mathbb{R}^\infty)$ spanned by $\mathscr{P}(\mathbb{R}^\infty)$ as follows

$$\mathscr{S} : \mathbb{R}\mathscr{P}(\mathbb{R}^\infty) \longrightarrow \mathrm{Mer}_{\mathrm{sep}}(\mathbb{C}^\infty), \quad \langle e_I \rangle \mapsto \prod_{i \in I} S(\varepsilon_i)$$

and

$$\mathscr{I} : \mathbb{R}\mathscr{P}(\mathbb{R}^\infty) \longrightarrow \mathrm{Mer}_{\mathrm{sep}}(\mathbb{C}^\infty), \quad \langle e_I \rangle \mapsto \prod_{i \in I} I(\varepsilon_i).$$

3.4 Evaluating separable meromorphic functions at poles

In order to çount"the number of lattice points

$$\text{"}\left(\sum_{\vec{n} \in \sum_{i \in I} \mathbb{Z}_{\geq 0} e_i} \vec{n}^{\vec{0}} \right)\text{"}$$

in the product cone $\langle e_I \rangle$ we want to evaluate $\mathscr{S}(\langle e_I \rangle)$ at $(\varepsilon_{i_1}, \cdots, \varepsilon_{i_k}) = \vec{0}$. Since $\mathscr{S}(\langle e_I \rangle) \in \mathrm{Mer}_{\mathrm{sep}}(\mathbb{C}^\infty)$ a first guess is to assign the value

$$\mathscr{S}_{\vec{0}}^{\mathrm{ren}} := \mathrm{ev}_{\vec{0}}^{\mathrm{ren}} \circ \mathscr{S}, \tag{12}$$

where $\mathrm{ev}_{\vec{0}}^{\mathrm{ren}}$ is defined in (11). Let us now describe the underlying algebraic framework, which might seem somewhat artificial in the rather trivial product cone situation. However, on the one hand even in this simple situation it is useful to control the "polar part"which one needs to extract in order to define the finite part, on the other hand it offers a good toy model to motivate otherwise relatively sophisticated techniques which can be generalized beyond product cones, namely to general convex cones [8].

[3] $\mathrm{Mer}_{\mathrm{sep}}(\mathbb{C}^\infty)$ is isomorphic to the filtered vector space $\mathscr{F} := \varinjlim(\mathrm{Mer}_0^1(\mathbb{C}))^n$ by assigning $f_1 \otimes \cdots \otimes f_n$ to $f_1(\varepsilon_1) \cdots f_n(e_n)$. But the map does not respect the tensor product. For example, $f \otimes f(\varepsilon_1, \varepsilon_e) = f(\varepsilon_1) f(\varepsilon_2) \neq f(\varepsilon_1)^2$.

3.5 A complement map on product cones

Let us first recall the properties of the set complement map.

Let $\mathscr{P}_f(E)$ be the set of finite subsets of a given set E equipped with the inclusion $\subseteq$ which defines a partial order compatible with the filtration of $\mathscr{P}_f(E)$ by the cardinal in the sense that $J \subseteq I$ implies $|J| \le |I|$. For $I \in \mathscr{P}_f(E)$ let

$$\mathfrak{s}(I) := \{J \in \mathscr{P}_f(E) \,|\, J \subseteq I\}$$

be the set of subsets of I.

The **set complement map** assigns to any $I \subseteq E$ a map

$$\begin{array}{rcl} \complement_I : \mathfrak{s}(I) & \longrightarrow & \mathfrak{s}(I) \\ J & \longmapsto & I \setminus J := I \cap \bar{J}. \end{array}$$

The complement $I \setminus J$ satisfies the following properties:

1. **Compatibility with the partial order:** Let $I, J \in \mathscr{P}_f(E)$ be such that $J \subseteq I$. Then for any $H \in \mathscr{P}_f(E)$ with $H \subseteq I \setminus J$ there exists unique $K \in \mathscr{P}_f(E)$; $J \subseteq K \subseteq I$ such that $H = I \setminus K$.
2. **Transitivity:** Let $I, J, K \in \mathscr{P}_f(E)$ be such that $K \subseteq J \subseteq I$. Then
$$(I \setminus K) \setminus (J \setminus K) = I \setminus J$$
3. **Compatibility with the filtration:** Let $I, J \in \mathscr{P}_f(E)$ be such that $J \subseteq I$. Then
$$\mathrm{card}(J) + \mathrm{card}(I \setminus J) = \mathrm{card}(I),$$
where card stands for the cardinality.

The set complement map on $\mathbb{Z}_{\ge 0}$ induces a complement map on the product cones. Let us first introduce some notations. Faces of the product cone $C := \langle e_I \rangle$ are of the form

$$F_J := \langle e_J \rangle$$

with $J \subset I$, each of them defining a cone with faces $F_{J'}$ where $J' \subset J$. The cone C therefore has $2^{|I|}$ faces, as many as subsets of I. The set $\mathscr{F}(C)$ of faces of the cone C is equipped with a partial order

$$F' \subset F \text{ if and only if } F' \text{ is a face of } F'$$

or equivalently, $F_{J'} \subset F_J$ if and only if $J' \subset J$. For $F' = F_{J'} \subset F = F_J$ we consider the complement set $\overline{F'}^F := F_{J \setminus J'}$, which again defines an element of $\mathscr{F}(C)$ and hence a cone. We define the complement map

$$\begin{array}{rcl} \mathscr{F}(C) & \longrightarrow & \mathscr{F}(C) \\ F_J & \longmapsto & \overline{F}_J^{\,C} = F_{I \setminus J}, \end{array} \tag{13}$$

which is an involution. As a consequence of the properties of the set complement map, it enjoys the following properties. Let $F \in \mathscr{F}(C)$.

1. **Compatibility with the partial order:** There is a one-to-one correspondence between the set of faces of C containing a given face F and the set of faces of the cone $\overline{F}^C$; for any face H of $\overline{F}^C$, there is a unique face G of C containing F such that $H = \overline{F}^G$.
2. **Transitivity:** $\overline{F}^C = \left(\overline{F'}^F\right)^{\overline{F'}^C}$ if $F' \subset F$.
3. **Compatibility with the filtration by the dimension:** For any face F of C we have $\dim(F) + \dim\left(\overline{F}^C\right) = \dim(C)$.

There is an alternative description of this complement map which is generalizable to general convex cones, those not necessarily obtained as product cones. For this we observe that for a face $F = F_J = \langle e_J \rangle$ of a product cone $C = \langle e_I \rangle$, we have

$$\overline{F}^C = F_{I \setminus J} = \pi_{F^\perp}(C), \tag{14}$$

where $F^\perp$ denotes the orthogonal space of the linear space spanned by the cone F in the linear space $\langle C \rangle$ spanned by C, and $\pi_{F^\perp}$ is the orthogonal projection from $\langle C \rangle$ onto $F^\perp$. Here the orthogonal projection is taken with respect to the canonical Euclidean product on $\mathbb{R}^\infty$. Eq. (14) follows from the fact that $\pi_{F^\perp}(e_i)$ is 0 for $i \in J$ and e_i for $i \notin J$.

3.6 Algebraic Birkhoff factorization on product cones

For each $i \ge 1$, the algebra $\mathscr{A}_i := \mathrm{Mer}_0(\mathbb{C}\varepsilon_i)$ is naturally isomorphic to $\mathscr{A} := \mathrm{Mer}_0(\mathbb{C})$ as the algebra of Laurent series. Following the minimal subtraction scheme we have a direct sum $\mathscr{A}_i = \mathscr{A}_{i,+} \oplus \mathscr{A}_{i,-}$ of two subalgebras $\mathscr{A}_{i,\pm} := \pi_\pm(\mathscr{A}_i)$. The maps $\mathscr{S}_i : \mathbb{R}\langle e_i \rangle \longrightarrow \mathrm{Mer}_0(\mathbb{C}\varepsilon_i)$ defined in Eq. (**??**) split accordingly $\mathscr{S}_i = \mathscr{S}_{i,+} + \mathscr{S}_{i,-}$ into a sum of maps $\mathscr{S}_{i,\pm} : \mathbb{R}\langle e_i \rangle \longrightarrow \mathscr{A}_{i,\pm}$.

We next consider separable functions in several variables. For disjoint subsets $I, J \subseteq \mathbb{N}$, define

$$\mathscr{A}_{I,+,J,-} := \left(\prod_{i \in I} \mathscr{A}_{i,+}\right) \left(\prod_{j \in J} \mathscr{A}_{j,-}\right).$$

Also denote $\mathscr{A}_{I,+,J,-} = \mathscr{A}_{I,+}$ if $J = \emptyset$. Then we have

$$\mathscr{A}_I := \prod_{i \in I} \mathscr{A}_i = \oplus_{I_1 \sqcup I_2 = I} \mathscr{A}_{I_1,+,I_2,-}\,.$$

Further denote

$$\mathscr{A}_{I,+} := \prod_{i \in I} \mathscr{A}_{i,+}, \quad \mathscr{A}_{I,-} := \prod_{J \subsetneq I} \mathscr{A}_{J,+,I \setminus J,-}$$

and

$$\mathscr{A}_\infty := \underset{\longrightarrow}{\mathrm{lím}}\, \mathscr{A}_I, \quad \mathscr{A}_{\infty,\pm} := \underset{\longrightarrow}{\mathrm{lím}}\, \mathscr{A}_{I,\pm}.$$

Thus we have

$$\mathscr{A}_\infty = \mathscr{A}_{\infty,+} \oplus \mathscr{A}_{\infty,-}.$$

$\mathscr{A}_{\infty,+}$ is a subalgebra but not $\mathscr{A}_{\infty,-}$. For example, $\mathscr{A}_{1,+}\mathscr{A}_{2,-}$ and $\mathscr{A}_{1,-}$ are in $\mathscr{A}_{\infty,-}$, but their product is not.

We want to generalize to product cones what we saw in the previous section, namely the fatct that the splitting $S = S_+ + S_-$ of the exponential sum corresponds to the Euler-Maclaurin formula $S = \mu + I$ with $\mu = S_+$.

So we need to generalize the minimal subtraction scheme and the Euler-Maclaurin formula to product cones. In the product cone framework, the minimal subtraction scheme generalizes to an elementary form of the more general algebraic Birkhoff factorization on coalgebras which we shall describe in the next section.

Proposition 2 *Given a product cone $C = \langle e_I \rangle$ in $\mathscr{P}(\mathbb{R}^\infty)$ the map $\mathscr{S}(C)$ extends to a meromorphic map in $\mathrm{Mer}_{\mathrm{sep}}(\mathbb{C}^\infty)$ with simple poles on the intersections of hyperplanes $\cap_{j\in J}\{\varepsilon_j = 0\}$ corresponding to faces $F_J = \langle e_I \rangle, J \subseteq I$ of the cone C. It decomposes as*

$$\mathscr{S}(C) = \sum_{F\in\mathscr{F}(C)} \mathscr{S}_+(\overline{F}^C)\mathscr{S}_-(F) \tag{15}$$

$$= \sum_{F\in\mathscr{F}(C)} \mu(\overline{F}^C)\mathscr{I}(F), \tag{16}$$

where (15) is known as algebraic Birkhoff factorization, (16) is known as Euler-Maclaurin formula and for a face $F = \langle e_K \rangle$ of the cone C, $\overline{F}^C = F_{I\setminus K}$ is the “complement face”defined in the previous paragraph and where we have set

$$\mathscr{S}(F) = \mathscr{S}(\langle e_K \rangle) := \prod_{i\in K} S(\langle e_i \rangle), \quad \mathscr{S}_\pm(F) = \mathscr{S}_\pm(\langle e_K \rangle) := \prod_{i\in K} \mathscr{S}_{i,\pm}.$$

Remark 3 *Eq. (15) which arises from the one-dimensional minimal substraction scheme can be viewed as a higher dimensional minimal subtraction scheme and Eq. (16) as a higher dimensional Euler-Maclaurin formula. When $k = 1$ they yield back the one dimensional minimal subtraction scheme and the Euler-Maclaurin formula applied to $S(\varepsilon)$.*

Proof. Let $C = \langle e_I \rangle$ for some finite subset I in $\mathbb{N}$. We have

$$\begin{aligned}
\mathscr{S}(C) &= \prod_{i\in I} \mathscr{S}_i(\langle e_i \rangle) \quad \text{(a product of sums)} \\
&= \prod_{i\in I} (\mathscr{S}_{i,+} + \mathscr{S}_{i,-})(\langle e_i \rangle) \quad \text{(a sum of products)} \\
&= \sum_{J\subset I} \left(\prod_{j\in I\setminus J} \mathscr{S}_{j,+}(\langle e_j \rangle) \right) \left(\prod_{j\in J} \mathscr{S}_{j,-}(\langle e_j \rangle) \right) \\
&= \sum_{F\in\mathscr{F}(C)} \mathscr{S}_+(\overline{F}^C)\mathscr{S}_-(F) \\
&= \sum_{F\in\mathscr{F}(C)} \mu(\overline{F}^C)\mathscr{I}(F).
\end{aligned}$$

■

The fact that the algebraic Birkhoff factorization (15) and the Euler-Maclaurin formula (16) coincide for product cones is a consequence of Eq. (9) which shows how, in the one dimensional case, the minimal subtraction scheme and the Euler-Maclaurin formula coincide for the exponential sum. The renormalized discrete sum in Eq. (12), which can be rewritten as

$$\mathscr{S}_{\vec{0}}^{\mathrm{reg}} = \mathrm{ev}_{\vec{0}} \circ \mathscr{S}_+ = \mathrm{ev}_{\vec{0}} \circ \mu,$$

is obtained from evaluating at zero the renormalized “holomorphic part”$\mathscr{S}_+$ of the exponential sum derived from the algebraic Birkhoff factorization (see (15)) or equivalently from evaluating at zero the renormalized interpolator μ derived from the Euler-Maclaurin formula (see (16)).

We have gone a long way around to recover our first guess (12). This approach using Birkhoff-Hopf factorization, even if somewhat artificial in the case of product cones, is nevertheless useful for it can be generalized to all rational polyhedral convex (lattice) cones [8] a case which will be briefly discussed at the end of the paper.

4. From complement maps to coproducts

We now set up an algebraic framework to derive an algebraic Birkhoff factorization from a complement map in a more general set up than the specific example of product cones which served as a toy model in the previous section.

4.1 Posets

Let $(\mathscr{P}, \leq)$ be a poset, i.e. a set $\mathscr{P}$ together with a partial order $\leq$. We do not assume that the poset is finite.

The poset is **filtered** if $\mathscr{P} = \bigcup_{n=0}^{\infty} \mathscr{P}_n$ with $\mathscr{P}_n \subset \mathscr{P}_{n+1}$. The degree of $A \in \mathscr{P}$ denoted by $|A|$ is the smallest integer n such that $A \in \mathscr{P}_n$. The partial order $\leq$ is compatible with the filtration if $A \leq B$ implies $|A| \leq |B|$.

We call a filtered poset $\mathscr{P}$ **connected** if $\mathscr{P}$ has a least element 1, called the **bottom** of $\mathscr{P}$, and we have $\mathscr{P}_0 = \{1\}$.

Example 4 *For a given set X (finite or infinite), the set $\mathscr{P}_f(X)$ of finite subsets of X equipped with the inclusion relation is a poset $(\mathscr{P}_f(X), \subseteq)$ filtered by the cardinal. It is connected since $\emptyset$ is the only subset of cardinal 0 and $\emptyset \subseteq A$ for any $A \in \mathscr{P}_f(X)$.*

Example 5 *This example can be regarded as a special case of the previous example but its pertinence for convex cones justifies that we treat it separately. The set $\mathscr{P}(\mathbb{R}^\infty) = \cup_{n=0}^{\infty} \mathscr{P}(\mathbb{R}^n)$ of closed product cones described in the previous section is filtered by the dimension and partially ordered by the partial order on the index sets. Equivalently, $F \leq C$ if the product cone F is a face of the product cone C. $\mathscr{P}(\mathbb{R}^\infty)$ is connected since the zero cone $\{0\}$ is the only cone of dimension 0 and $\{0\} \leq C$ for any $C \in \mathscr{P}(\mathbb{R}^\infty)$ as 0 is a vertex of any product cone.*

4.2 Complement maps on posets

Definition 6 *Let $(\mathscr{P}, \leq)$ be a poset such that for any $E \in \mathscr{P}$*

$$\mathfrak{s}(E) := \{A \in \mathscr{P} \,|\, A \leq E\} \tag{17}$$

is a finite set. A **complement** *map on $\mathscr{P}$ assigns to any element $E \in \mathscr{P}$ a map*

$$\begin{array}{rcl} \complement_E : \mathfrak{s}(E) & \longrightarrow & \mathscr{P} \\ A & \longmapsto & E \backslash A \end{array}$$

satisfying the following properties

1. **Compatibility with the partial order:** *Let A, C in $\mathscr{P}$ be such that $A \leq C$. Then*

$$\mathfrak{s}(C \backslash A) = \{B \backslash A \,|\, A \leq B \leq C\}.$$

2. **Transitivity:** *Let A, B, C in $\mathscr{P}$ be such that $A \leq B \leq C$. Then*

$$(C \backslash A) \setminus (B \backslash A) = C \backslash B.$$

3. **Compatibility with the filtration:** *Assume that the poset is filtered: $\mathscr{P} = \cup_{n \in \mathbb{N}} \mathscr{P}_n$. Then the complement map is compatible with the filtration in the sense that*

$$A \leq C \Longrightarrow |C \backslash A| = |C| - |A|.$$

4. **Compatibility with the bottom:** *Assume that the poset is connected and let $\mathscr{P}_0 = \{1\}$. Then*

$$C \backslash 1 = C \quad \text{for all } C \in \mathscr{P}.$$

Condition (d) is obviously satisfied by previous examples of complement maps.

Example 7 *Let E be a set. For $X \in \mathscr{P}_f(E)$, the complement set map:*

$$\begin{array}{rcl} \mathscr{P}_f(X) & \longrightarrow & \mathscr{P}_f(X) \\ Y & \longmapsto & X \setminus Y := X \cap \overline{Y} \end{array}$$

defines a complement map compatible with the filtration by the dimension.

Example 8 *As we saw in the previous section, the set complement map on $\mathbb{Z}_{\geq 0}$ induces a complement map on product cones which we recall here for convenience. Given a product cone $\langle e_I \rangle$ and a subset $J \subseteq I$, the map $\langle e_J \rangle \longmapsto \langle e_{I \backslash J} \rangle$ defines a complement map on $\mathscr{P}(\mathbb{R}^\infty)$ compatible with the filtration by the dimension of the cone.*

4.3 A complement map on general convex cones

We now generalize the complement map built on product cones to general convex cones by means of an orthogonal projection.

Let $\mathscr{F}(C)$ be the set of all faces of a convex cone $C \subseteq \mathbb{R}^k$. We borrow the following concept from [2] (see also [8]) which we refer the reader to for further details. The **transverse cone** to $F \in \mathscr{F}(C)$ is

$$t(C, F) := (C + \mathrm{lin}(F))/\mathrm{lin}(F), \tag{18}$$

(where lin stands for the linear span) which we identify to the cone in $\mathscr{C}(\mathbb{R}^\infty)$ defined by the projection $\pi_{F^\perp}(C)$ of C onto the orthogonal complement[4] $\mathrm{lin}(F)^\perp$ in $\mathrm{lin}(C)$ for the canonical scalar product on $\mathbb{R}^\infty$.

Example 9 *The transverse cone to a face $F = \langle e_J \rangle$ of a product cone $\langle e_I \rangle$ is the cone $\langle e_{I \backslash J} \rangle$, which corresponds to the transverse cone $t(\langle e_I \rangle, \langle e_J \rangle)$.*

Example 10 *The transverse cone to the face $F = \langle e_1 + e_2 \rangle$ in the cone $C = \langle e_1, e_1 + e_2 \rangle$ is the cone $t(C, F) = \langle e_1 - e_2 \rangle$. Note that $t(C, F)$ is not a face of C.*

The subsequent lemma follows from the fact that the orthogonal complement on subspaces fulfills the properties of a complement map.

Lemma 11 *The map*

$$\begin{array}{rcl} \mathscr{F}(C) & \longrightarrow & \mathscr{C}(\mathbb{R}^\infty) \\ F & \longmapsto & t(C, F) \end{array}$$

which to a face F of a cone C assigns the transverse cone *$t(C, F)$, is a complement map. More precisely, it enjoys the following properties.*

1. **Compatibility with the partial order:** *The set of faces of the cone $t(C, F)$ equals*

$$\{t(G, F) \,|\, G \text{ a face of } C \text{ containing } F\}$$

.

2. **Transitivity:** *$t(C, F) = t(t(C, F'), t(F, F'))$ if F' is a face of F.*

3. **Compatibility with the dimension filtration:** $\dim(C) = \dim(F) + \dim(t(C, F))$ *for any face F of C.*

4. **Compatibility with the bottom:** *$\mathscr{C}$ is connected since $\mathscr{C}_0$ is reduced to $1 := \{0\}$ and for any cone C we have $t(C, \{0\}) = C$.*

---

[4] Our approach, like the one of Berline and Vergne in [2], actually requires a choice a rational lattice which consists of a pair built from a cone and a rational lattice in the linear space spanned by the cone. We refer the reader to [8] for a detailed description.

4.4 Coproducts derived from complement maps

Loosely speaking, coalgebras are objects dual to algebras. More precisely, algebras are dual to coalgebras but the converse only holds in finite dimensions (see e.g. [3]).

Definition 12 *A* **(counital) coalgebra** *is a linear space $\mathscr{C}$ (here over $\mathbb{R}$) equipped with two linear maps:*

1. *a* **comultiplication** $\Delta : \mathscr{C} \to \mathscr{C} \otimes C$ *written in Sweedler's notation [16]*

$$\Delta c = \sum_{(c)} c_{(1)} \otimes c_{(2)},$$

which is **coassociative**

$$(I \otimes \Delta) \otimes \Delta = (\Delta \otimes I) \otimes \Delta.$$

The coassociativity of Δ translates to the following commutative diagram

$$\begin{tikzcd} \mathscr{C} \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & \mathscr{C} \otimes \mathscr{C} \arrow[d, "I\otimes\Delta"] \\ \mathscr{C} \otimes \mathscr{C} \arrow[r, "\Delta\otimes I"] & \mathscr{C} \otimes \mathscr{C} \otimes \mathscr{C} \end{tikzcd}$$

and can be expressed in the following compact notation:

$$\sum_{(c)} c_{(1)} \otimes \left(\sum_{(c_{(2)})} (c_{(21)}) \otimes (c_{(22)}) \right) = \sum_{(c)} \left(\sum_{(c_{(1)})} (c_{(11)}) \otimes (c_{(12)}) \right) \otimes c_{(2)}.$$

With Sweedler's notation [16], both these expressions read

$$\sum_{(c)} c_{(1)} \otimes c_{(2)} \otimes c_{(3)}.$$

2. *a* **counit** $\varepsilon : \mathscr{C} \to \mathbb{R}$ *satisfying the* **counitarity** *property*

$$(I_{\mathscr{C}} \otimes \varepsilon) \circ \Delta = (\varepsilon \otimes I_{\mathscr{C}}) \circ \Delta = I_{\mathscr{C}}, \qquad (19)$$

with the identification $\mathscr{C} \otimes \mathbb{R} \simeq \mathscr{C} \simeq \mathbb{R} \otimes \mathscr{C}$. This translates to the following commutative diagram:

$$\begin{tikzcd} \mathscr{C} \otimes \mathscr{C} \arrow[d, "\varepsilon\otimes I_{\mathscr{C}}"'] & \mathscr{C} \arrow[l, "\Delta"'] \arrow[r, "\Delta"] \arrow[d, "\varepsilon"] & \mathscr{C} \otimes \mathscr{C} \arrow[d, "I_C\otimes\varepsilon"] \\ \mathbb{R} \otimes \mathscr{C} \arrow[r, "\cong"] & \mathscr{C} & \mathscr{C} \otimes \mathbb{R} \arrow[l, "\cong"'] \end{tikzcd}$$

The fact that ε is a counit can be expressed by means of the following formula

$$c = \sum_{(c)} \varepsilon(c_{(1)}) c_{(2)} = \sum_{(c)} c_{(1)} \varepsilon(c_{(2)}).$$

The coalgebra is **cocommutative** *if $\tau \circ \Delta = \Delta$ where $\tau : \mathscr{C} \otimes \mathscr{C} \to \mathscr{C} \otimes \mathscr{C}$ is the flip $c_1 \otimes c_2 \longmapsto c_2 \otimes c_1$. This translates to the following commutative diagram:*

$$\begin{tikzcd} & \mathscr{C} \arrow[dl, "\Delta"'] \arrow[dr, "\Delta"] & \\ \mathscr{C} \otimes \mathscr{C} \arrow[rr, "\tau"] & & \mathscr{C} \otimes \mathscr{C} \end{tikzcd}$$

and the equation

$$\sum_{(c)} c_{(1)} \otimes c_{(2)} = \sum_{(c)} c_{(2)} \otimes c_{(1)}.$$

The coalgebra $\mathscr{C}$ is **coaugmented** *if there is a morphism of coalgebras $u : \mathbb{R} \to \mathscr{C}$ in which case we have $\varepsilon \circ u = I_{\mathbb{R}}$ and we set $1_C := u(1_{\mathbb{R}})$ where $1_{\mathbb{R}}$ is the unit in $\mathbb{R}$. If $\mathscr{C}$ is coaugmented, then $\mathscr{C}$ is canonically isomorphic to Ker $\varepsilon \oplus \mathbb{R}1_{\mathscr{C}}$. The kernel Ker ε is often denoted by $\overline{\mathscr{C}}$ so $\mathscr{C} = \overline{\mathscr{C}} \oplus \mathbb{R}1_{\mathscr{C}}$. Let $\mathscr{C} = \mathbb{R}1_{\mathscr{C}} \oplus \overline{\mathscr{C}}$ be a coaugmented coalgebra. The coradical filtration on $\mathscr{C}$ is defined as follows: Define $\mathscr{F}_0\mathscr{C} := \mathbb{R}1_{\mathscr{C}}$, and for $r \in \mathbb{N}$, we set*

$$\mathscr{F}_r\mathscr{C} := \mathbb{R}1 \oplus \{x \in \overline{\mathscr{C}} \,|\, \overline{\Delta}^n x = 0 \quad \forall n > r\}.$$

Here we have set $\overline{\Delta}x = \Delta x - (1_C \otimes x + x \otimes 1_C)$ and $\overline{\Delta}^n$ is the n-th iteration. A coalgebra $\mathscr{C}$ is said to be **conilpotent** *(or sometimes* **connected** *in the literature) if it is coaugmented and if the filtration is exhaustive, that is $\mathscr{C} = \cup_{r\in\mathbb{N}} \mathscr{F}_r\mathscr{C}$.*

Coproducts typically arose from complement maps.

Proposition 13 *Let a poset $(\mathscr{P}, \leq)$ be such that for any E in $\mathscr{P}$ the set $\mathfrak{s}(E)$ defined as in Eq. (17) is finite and let it be equipped with a complement map, which assigns to any element $E \in \mathscr{P}$ a map*

$$\complement_E : \mathfrak{s}(E) \longrightarrow \mathscr{P}$$

$$A \longmapsto E \backslash A.$$

Then the map

$$\begin{array}{rcl} \Delta : \mathscr{P} & \longrightarrow & \mathscr{P} \otimes \mathscr{P} \\ E & \longmapsto & \sum_{A \in \mathfrak{s}(E)} E \backslash A \otimes A, \end{array}$$

extends linearly to a coassociative coproduct on the space $\mathbb{K}\mathscr{P}$ freely generated over a field $\mathbb{K}$ by $\mathscr{P}$.

If the poset is filtered $\mathscr{P} = \cup_{n\in\mathbb{N}} \mathscr{P}_n$ and the complement map is compatible with the filtration then so is the coproduct, that is, if C is in $\mathscr{P}_n$, then ΔC is in $\sum_{p+q=n} \mathscr{P}_p \otimes \mathscr{P}_q$.

Let $\varepsilon : \mathscr{P} \to \mathbb{K}$ be zero outside $\mathscr{P}_0$ where it takes the value one and let us denote its linear extension to $\mathbb{K}\mathscr{P}$ by the same symbol. If moreover the poset $\mathscr{P}$ is connected, then the linear space $(\mathbb{K}\mathscr{P}, \Delta, \varepsilon)$ is a counital connected coalgebra.

Proof. The coassociativity and the counitarity easily follow from the properties of the complement map. ■

Example 14 *The vector space $\mathbb{R}\mathscr{P}_f(E)$ spanned by finite subsets of a finite set E defines a conilpotent coalgebra.*

Example 15 *The free algebra $\mathbb{R}\mathscr{C}(\mathbb{R}^\infty)$ spanned by closed convex cones pointed at zero in $\mathbb{R}^\infty$ defines a conilpotent coalgebra.*

5. Algebraic Birkhoff factorization on a conilpotent coalgebra

We give a generalization ([8]) of the algebraic Birkhoff factorization used for renormalization purposes in quantum field theory (see [4, 14]) in so far as we weaken the assumptions on the source space which is not anymore assumed to be a Hopf algebra but only a coalgebra, as well as on the target algebra which is not anymore required to decompose into two subalgebras. We first define the convolution product and give its main properties.

5.1 The convolution product

Let $(\mathscr{A}, m_{\mathscr{A}}, 1_{\mathscr{A}})$ be an (unital) commutative algebra over $\mathbb{R}$. We quote the following result from [14, Proposition II.3.1].

Proposition 16 *Let $(\mathscr{C}, \Delta_{\mathscr{C}}, \varepsilon_{\mathscr{C}})$ be a (counital) coalgebra over $\mathbb{R}$.*

1. *The* **convolution** *product on $\mathscr{L}(\mathscr{C}, \mathscr{A})$ defined as*

$$\phi * \psi = m_{\mathscr{A}} \circ (\phi \otimes \psi) \circ \Delta_{\mathscr{C}}$$

is associative. In Sweedler's notation it reads:

$$\phi * \psi(x) = \sum_{(x)} \phi(x_{(1)})\psi(x_{(2)}).$$

2. *$e := u_{\mathscr{A}} \circ \varepsilon_{\mathscr{C}}$ is a unit for the convolution product on $\mathscr{L}(\mathscr{C}, \mathscr{A})$.*

Example 17 *The convolution product of two maps ϕ and ψ in $\mathscr{L}(\mathscr{P}(\mathbb{R}^\infty), \mathscr{A})$ on a product cone $C = \langle e_I \rangle$ derived from the complement map described in Example 8 reads*

$$\phi * \psi(\langle e_I \rangle) = \sum_{J \subset I} \phi\left(\langle e_{I \setminus J} \rangle\right) \psi(\langle e_J \rangle) = \sum_{F \in \mathscr{F}(C)} \phi(\overline{F}^C)\, \psi(F)$$

with the notation of Eq. (14).

Setting $\mathscr{A} = Mer_{\mathrm{sep}}(\mathbb{C}^\infty)$, then Eqs. (15) and (16) seen as identities of maps on product cones read

$$\mathscr{S} = \mathscr{S}_+ * \mathscr{S}_- = \mu * \mathscr{I}. \tag{20}$$

We refer to [14, Proposition II.3.1.]for teh subsequent assertion.

Proposition 18 *Let $\mathscr{C}$ be a connected augmented coalgebra and $\mathscr{A}$ an algebra. The set*

$$\mathscr{G}(\mathscr{C}, \mathscr{A}) := \{\phi \in \mathscr{L}(\mathscr{C}, \mathscr{A}), \quad \phi(1_{\mathscr{C}}) = 1_{\mathscr{A}}\}$$

endowed with the convolution product is a group with unit $e := \varepsilon_{\mathscr{C}} \circ u_{\mathscr{A}}$ and inverse

$$\phi^{*(-1)}(x) := \sum_{k=0}^{\infty} (e - \phi)^{*k}(x) \tag{21}$$

is well defined as a finite sum.

Example 19 *Back to Example 17, we can rewrite the renormalized holomorphic part (20) of S as*

$$\mathscr{S}_+ = \mathscr{S} * \mathscr{S}_-^{*(-1)} = \mathscr{S} * \mathscr{I}^{*(-1)}.$$

5.2 Algebraic Birkhoff factorization

We quote the following result which can be proved as in [8, Theorem 3.2] ignoring the differential structure discussed there.

Theorem 20 *Let $\mathscr{C} = \bigoplus_{n \geq 0} \mathscr{C}^{(n)}$ be a connected coalgebra and let $\mathscr{A}$ be a unitary algebra. Let $\mathscr{A} = \mathscr{A}_1 \oplus \mathscr{A}_2$ be a linear decomposition such that $1_{\mathscr{A}} \in \mathscr{A}_1$ and let P be the induced projection onto $\mathscr{A}_1$ parallel to $\mathscr{A}_2$.*
Given $\phi \in \mathscr{G}(\mathscr{C}, \mathscr{A})$, we define two maps $\varphi_i \in \mathscr{G}(\mathscr{C}, \mathscr{A}), i = 1,2$ defined by the following recursive formulae on $\ker \varepsilon$:

$$\varphi_1(x) = -P\left(\varphi(x) + \sum_{(x)} \varphi_1(x')\varphi(x'')\right), \tag{22}$$

$$\varphi_2(x) = (\mathrm{id}_A - P)\left(\varphi(x) + \sum_{(x)} \varphi_1(x')\varphi(x'')\right), \tag{23}$$

where, following Sweedler's notation, we have set $\overline{\Delta}x = \sum x' \otimes x''$.

1. *We have $\varphi_i(\ker \varepsilon) \subseteq \mathscr{A}_i$ and hence $\varphi_i : \mathscr{C} \to \mathbb{K}1_{\mathscr{A}} + \mathscr{A}_i$. Moreover, the following factorization holds*

$$\varphi = \varphi_1^{*(-1)} * \varphi_2. \tag{24}$$

2. *$\varphi_i, i = 1,2$, are the unique maps in $\mathscr{G}(\mathscr{C}, \mathscr{A})$ such that $\varphi_i(\ker \varepsilon) \subseteq \mathscr{A}_i$ for $i = 1,2$, and satisfying Eqn. (24).*

3. *If moreover $\mathscr{A}_1$ is a subalgebra of $\mathscr{A}$ then $\phi_1^{*(-1)}$ lies in $\mathscr{G}(\mathscr{C}, \mathscr{A}_1)$.*

6. Algebraic Birkhoff factorization on cones

The algebraic Birkhoff factorization can be carried out from exponential sums on product cones to exponential sums on general convex polyhedral cones using the complement map described in Lemma 11 built from the transverse cone to a face [8], which generalizes the orthogonal complement used in the case of product cones. We first need a coproduct on cones.

6.1 From the transverse cone to a coproduct on cones

Here we consider both closed convex (polyhedral) cones in $\mathbb{R}^k$

$$\langle v_1, \cdots, v_n\rangle := \mathbb{R}\{v_1, \cdots, v_n\} = \mathbb{R}_{\geq 0} v_1 + \cdots + \mathbb{R}_{\geq 0} v_n,$$

where $v_i \in \mathbb{R}^k$, $i = 1, \cdots, n$ defined previously and open cones defined in a similar manner replacing $\mathbb{R}_{\geq 0}$ by $\mathbb{R}_+$. Product cones $\langle e_i, i \in I\rangle$ with $I \subset \{1, \cdots, k\}$ and $\{e_i \,|\, i \in \{1, \cdots, k\}\}$ the canonical basis of $\mathbb{R}^k$ are convex cones. We shall focus here on Chen cones $\langle e_{i_1}, e_{i_1} + e_{i_2}, \cdots, e_{i_1} + \cdots + e_{i_n}\rangle$ with $\{i_1, \cdots, i_n\} \subset \{1, \ldots, k\}$, which are closed convex cones as well as their open counterparts.

Both the complement map defined by means of the transverse map in Lemma 11 and the corresponding coproduct defined in Proposition 13 (see also Example 15) are compatible with subdivisions in a suitable sense. Recall that a **subdivision** of a cone C is a set $\{C_1, \cdots, C_r\}$ of cones such that

(i) $C = \cup_{i=1}^r C_i$,

(ii) $C_1, \cdots, C_r$ have the same dimension as C and

(iii) $C_1, \cdots, C_r$ intersect along their faces, i.e., $C_i \cap C_j$ is a face of both C_i and C_j.

Example 21 *The product cone $\langle e_1, e_2\rangle$ can be subdivided into two Chen cones $\langle e_1, e_1 + e_2\rangle$ and $\langle e_1 + e_2, e_2\rangle$.*

6.2 The integeration and summation maps on general convex cones

To a simplicial convex (closed) cone $C \subset \mathbb{Z}^k$, namely one whose generators are linearly independent, one can assign an exponential sum and an exponential integral which can informally be described as follows

$$S^c(C)(\vec{\varepsilon}) := \sum_{\vec{n} \in C \cap Z^k} e^{\langle \vec{n}, \vec{\varepsilon}\rangle}; \quad S^o(C)(\vec{\varepsilon}) := \sum_{\vec{n} \in C^o \cap \mathbb{Z}^k} e^{\langle \vec{n}, \vec{\varepsilon}\rangle}; \quad I(C)(\vec{\varepsilon}) = \int_C e^{\langle \vec{x}, \vec{\varepsilon}\rangle}\, d\vec{x}.$$

Here C^o is the open cone given by the interior of C and ε is taken in

$$\check{C}_k^- := \left\{ \vec{\varepsilon} := \sum_{i=1}^k \varepsilon_i e_i^* \,\middle|\, \langle \vec{x}, \vec{\varepsilon}\rangle < 0 \text{ for all } \vec{x} \in C \right\},$$

where $\{e_i^* \,|\, i \in \{1, \ldots, k\}\}$ is the dual canonical basis and $\langle \vec{x}, \vec{\varepsilon}\rangle$ the natural pairing $R^k \otimes \left(\mathbb{R}^k\right)^* \to \mathbb{R}$.

Remark 22 *A precise formulation would require introducing a lattice attached to the cone, so considering lattice cones instead of mere cones (see [8]). This then extends to any convex cones by additivity on subdivisions.*

Whereas exponential sums on product cones take their values on products of meromorphic functions in one variable, exponential sums on general convex cones take their values in the larger space of meromorphic maps with simple linear poles supported by the faces of the cone.

6.3 Meromorphic functions with linear poles

In order to implement a Birkhoff-Hopf factorisation on the summation map, we need to define a projection π_+ onto the holomorphic part of a meromorphic function with linear poles, which we do here somewhat informally. A precise definition can be found in [9] where we show that a meromorphic function $f = \frac{h}{L_1 \cdots L_n}$ on $\mathbb{C}^k$ with linear poles $L_i, i = 1, \cdots, n$ given by linear forms and h a holomorphic function at zero, uniquely decomposes as

$$f = \sum_{i=1}^n \left(\frac{h_i(\vec{\ell}_i)}{\vec{L}_i^{\vec{s}_i}} + \phi_i(\vec{\ell}_i, \vec{L}_i) \right), \tag{25}$$

with $|\vec{s}_i| > 0$ and where $\vec{L}_i = (L_{i1}, \cdots, L_{im_i})$, $\{L_{i1}, \cdots, L_{im_i}\}$ is a linear independent subset of $\{L_1, \cdots, L_n\}$, extended to a basis $\{\vec{L}_i, \vec{\ell}_i\}$ of $\mathbb{C}^k$, with $\vec{\ell}_i = (\ell_{i(m_i+1)}, \cdots \ell_{ik})$, L_{ij}, ℓ_{im} orthogonal for the canonical inner product on $\mathbb{C}^k$ and $h_i(\vec{\ell}_i)$ holomorphic (reduced to a constant when $k = 1$). Then we call $f_+ := \pi_+(f) = \sum_{i=1}^n \phi_i$, which is a germ of holomorphic function in the independent variables $\vec{\ell}_i$ and $\vec{L}_i$, the **holomorphic part** of f and $f_- := (1 - \pi_+)(f) = \sum_{i=1}^n \frac{h_i(\vec{\ell}_i)}{\vec{L}_i^{\vec{s}_i}}$ the **polar part** of f.

In order to discuss examples, it is convenient to set the following notation. Given k linear forms $L_1, \cdots, L_k$, we set

$$[L_1, \cdots, L_k] := \frac{e^{L_1}}{1 - e^{L_1}} \frac{e^{L_1 + L_2}}{1 - e^{L_1 + L_2}} \cdots \frac{e^{L_1 + L_2 + \cdots + L_k}}{1 - e^{L_1 + L_2 + \cdots + L_k}}. \tag{26}$$

So, for any (closed) Chen cone $C_k = \langle e_1, e_1 + e_2, \cdots, e_1 + \cdots + e_k\rangle$ (here $e_1, \ldots, e_k$ is the canonical basis of $\mathbb{R}^k$), we have

$$S^o(C_k)(\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_k) = [\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_k].$$

Example 23 *1. Take $k = 1$. Let $f(\varepsilon) = \frac{e^\varepsilon}{1 - e^\varepsilon} = \frac{1}{e^{-\varepsilon} - 1} = -\frac{\mathrm{Td}(-\varepsilon)}{-\varepsilon}$ on $\mathbb{C}$. Then by Eq. (5) we have*

$$f(\varepsilon) = -\frac{1}{\varepsilon} - \frac{1}{2} - \sum_{k=1}^K \frac{B_{2k}}{(2k)!} \varepsilon^{2k-1} + o(\varepsilon^{2K}) = -\frac{1}{\varepsilon} + \phi(\varepsilon), \tag{27}$$

with

$$\phi(\varepsilon) := -\frac{1}{2} - \sum_{k=1}^K \frac{B_{2k}}{(2k)!} \varepsilon^{2k-1} + o(\varepsilon^{2K}) = -\frac{1}{2} - \frac{1}{12}\varepsilon + \frac{1}{720}\varepsilon^3 + \cdots \tag{28}$$

holomorphic at zero so $\pi_+(f) = \phi(\varepsilon)$.

2. *Let $k=2$ and let $f(\varepsilon)=[\varepsilon_1,\varepsilon_1+\varepsilon_2]$. Applying Eq. (27) we write*

$$\begin{aligned}
\pi_+([\varepsilon_1,\varepsilon_2]) &= \pi_+\left(\begin{array}{c}\left(-\frac{1}{\varepsilon_1}+\phi(\varepsilon_1)\right)\cdot\\ \left(-\frac{1}{\varepsilon_1+\varepsilon_2}+\phi(\varepsilon_1+\varepsilon_2)\right)\end{array}\right)\\
&= \pi_+\left(\begin{array}{c}-\frac{\phi(\varepsilon_1+\varepsilon_2)}{\varepsilon_1}-\frac{\phi(\varepsilon_1)}{\varepsilon_1+\varepsilon_2}+\\ \phi(\varepsilon_1)\phi(\varepsilon_1+\varepsilon_2)\end{array}\right)\\
&= -\frac{\phi(\varepsilon_1+\varepsilon_2)-\phi(\varepsilon_2)}{\varepsilon_1}-\\
&\quad \frac{\phi(\varepsilon_1)-\phi\left(\frac{\varepsilon_1-\varepsilon_2}{2}\right)}{\varepsilon_1+\varepsilon_2}+\\
&\quad \phi(\varepsilon_1)\phi(\varepsilon_1+\varepsilon_2).
\end{aligned}$$

So, just as in the one-dimensional case, a meromorphic map function with linear poles also decomposes as a sum of a holomorphic part and a polar part. We shall denote by π_+ the corresponding projection onto the holomorphic part.

6.4 Algebraic Birkhoff factorization for discrete sums on general convex cones

Consequently, one can implement an algebraic Birkhoff factorization [8] on the coalgebra of convex polyhedral cones. [5] Just as the algebraic Birkhoff factorization gave rise to an Euler-Maclaurin formula on product cones, when the inner product used to defined the coproduct on cones coincides with the inner product used to decompose the space of meromorphic germs, the algebraic Birkhoff factorization of the exponential sum on a convex (lattice) cone yields back Berline and Vergne's local Euler-Maclaurin formula [2]. To prove this identification which is easy to see on smooth cones, we subdivide a general convex cone into simplicial ones and use the compatibility of S_- in the factorization procedure with subdivisions. This compatibility is shown by means of a rather involved combinatorial proof.

The “holomorphic part.ºf the exponential discrete sum on the one dimensional cone $[0,+\infty[$ generates renormalized zeta values at non-positive integers as coefficients of its Taylor expansion at zero, a fact that we have left out in these notes. Similarly, one can show [8] that the ”holomorphic part.ºf the exponential discrete sums on general convex polyhedral cones obtained from an algebraic Birkhoff factorization, generates what we call renormalized *conical* zeta values at non-positive integers which arise as coefficients of its Taylor expansion at zero. It turns out that the ”holomorphic part.ºf the exponential sums $S^c(C)$ and $S^o(C)$ on a cone C derived from the algebraic Birkhoff factorization actually coincides with the projection $\pi_+(S^c(C))$ and $\pi_+(S^o(C))$, when the inner product used to defined the coproduct on cones coincides with the inner product used to decompose the space of meromorphic germs respectively, onto their holomorphic part when seen as meromorphic functions with linear poles.

[5] We actually carry out the algebraic Birkhoff factorization on lattice cones.

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