ON THE MODULAR REPRESENTATIONS OF GROUP WITH p-COMPLEMENT

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ABSTRACT
Let G be a finite group containing a p-complement and Let R a complete discrete valuation ring with ideal maximal (P) and residue field k = R/(P) of characteristic p. In this paper we show that the regular representation can be decomposed as direct sum of tensor product of modules. Moreover, we prove that G satisfies the Alperin's weight conjecture in characteristic p.

RESUMEN
Sea el grupo finito G, con p complemento, y sea el anillo R una discreta valuación completa, con campo residual k de característica p. En este trabajo se demuestra que la representación regular RG se descompone como una suma directa de productos tensoriales de módulos. Además se verifica que G satisface la Conjetura de Alperin en característica p.

1. INTRODUCTION

A Hall divisor of an integer n is a divisor d of n such that d and n/d are coprime. For example, the Hall divisors of 24 are 1, 3, 8, and 24.

A Hall subgroup of a finite group G is a subgroup whose order is a Hall divisor of the order of G. If π is a set of primes, then a Hal π-subgroup is a subgroup whose order is a product of primes in π, and whose index is not divisible by any primes in π.

If p is a prime number, we write p´ for the set of all primes not equal to p. A Hall p-subgroup of a finite group G is called a p-complement.

Let G be a finite group containing a p-complement and (F, R, k) ap-modular sys-tem. In the present paper are given descriptions of the blocks of finite groups with p-complement. We show that G satisfies the Alperin's weight conjecture. The paper is structured as follows. Section 2 contains some results in modular representation theory of finite groups. In the section 3 we show that the indecomposable projective modules can be expressed as a tensor product of modules. In section 4 we verify the Alperin's weight conjecture. In section 5 we verify the Alperin's weight conjecture in block form.

1.1 Notations and Definitions

Let G be a finite group, p be a prime divisor of $|G|$ and R be a complete discrete valuation ring with quotient field F of characteristic 0. We assume that the residue field k = R/(π) has characteristic p, where (π) denotes the unique maximal ideal of R. With this assumption we refer to the triple (F; R; k) as a splitting p-modular system, and J(G) denotes the Jacobson radical of this ring of RG. We denote the largest normal p-subgroup of G by $O_p(G)$, and denote the´p-core of G by $O_{p'}(G)$. We denote a p-complement of G by $H_{p'}(G)$. The factor group G = $G/O_p(G)$ is called reduced group modulo $O_p(G)$. We denote the subgroup $O_p(G) \times H_{p'}(G)$ of G by $\tilde{G}$. If H is any subgroup of G then [G/H] denotes the set of representatives of left cosets gH.

2. PRELIMINARY

We consider the surjection $RG \to k\tilde{G}$. We denote the kernel of the surjection by $I_p(G) \subseteq J(G)$.

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$$RG/I_p(G) \cong k\tilde{G} \qquad (2.0.1)$$

Lemma 2.0.1. Let G be a finite group. Then the simple RG-modules are precisely the simple $k\tilde{G}$-modules made in to RG-modules via the surjection RG $\to$ $k\tilde{G}$.

Proof. If S is a simple $k\tilde{G}$-module, then also S is a simple $k\tilde{G}$-module, since RG/Ip = (G) $\cong$ $k\tilde{G}$ and Ip(G) annihilates the simple RG-modules.

Recall that if p is a prime, then an element in a finite group is said to be p-regulari fith as order prime top.

Proposition 2.0.2. Let G be a finite group. Then the number of non-isomorphic simple RG-modules equals the number of conjugacy classes of p-regular elements of the reduced group $\overline{G}$.

Proof. It well known that the number of non-isomorphic simple $k\tilde{G}$-modules equals the number of conjugacy classes of p-regular elements of G(See [3] theorem 9.11).There sult follows by (2.0.1). □

Proposition 2.0.3. Let G be a finite group.

1. For each simple RG-module S there is an indecomposable projective kG-module P_S = kGe. Here e is a primitive idempotent in kG such thate S $\neq$ 0.

2. For each simple RG-module S there is an indecomposable projective RG-module $\overline{P}_S = RG\hat{e}$ with the property that $\hat{P}_S/(\Pi)\hat{P}_S \cong P_S$ is the projective cover of S as a kG-module. Here $\hat{e}$ is a primitive idempotentin RG Duch that $\hat{e} \neq 0$.

Proof.

1. As k is Artinian ring and kG is a k-algebra finitely generated as k-vector space, it is Artinian. Therefore kG can be expressed as the direct sum of indecomposable projective kG-modules. Then we can write $1 = e_1 + \ldots + e_n$ in kG, where the e_i are primitive orthogonal idempotents. Hence for each simple RG-module S there is a unique summand e_i such that $e_iS \neq 0$. We define $P_S = kGe_i$. Since e_i is a primitive idempotent the result follows.

2. We omit the proof (See (3(proposition (9.15)).

Proposition 2.0.4. Let G be a finite group

1. Every finitely-generated indecomposable projective kG-module M is isomorphic to P_S for some simple module S.

2. Every finitely-generated indecomposable projective RG-module $\hat{M}$ is isomorphic to $\hat{P}_S$ for some simple module S.

Proof.

1. As kG is Artinian ring and M is finitely-generated indecomposable projective, it is Artinian. Hence the radical quotient M/Rad(M) $\cong$ S is a simple kG-module. By last proposition we have

$$M/Rad(M) \cong PS/Rad(PS) \cong S$$

   By Nakayama's lemma (See (3 (theorem 7.6) the modules M and PS are projective covers of the irradical quotients, so M $\cong$ PS by uniqueness of projective covers (See (3 (proposition 7.8).

2. We omit the proof (See [3] proposition 9.15 part 1).

3. QUASI-SIMPLE AND PROJECTIVE MODULES

Let G be a finite group with p-complement $H_{p'}(G)$. Observe that the reduced group $\overline{G}$ also has a p-complement $H_{p'}(\overline{G})$ which is isomorphic to $H_{p'}(G)$. A minimal p-subgroup H of G relative to which the indecomposable module U is projective is called a vertex of U, and it is defined up to conjugacy in G.

Theorem 3.0.5. Let G be a finite group with p-complement $H_{p'}(G)$. Set $P \in Syl_p(G)$ and $\overline{P} = P/O_p(G) \in Syl_p(\overline{G})$. Assume that $\overline{S}_1,..., \overline{S}_n$ is a complete list of non-isomorphic simple $RH_{p'}(\overline{G})$-modules.

1. If $Stab_{(\overline{G})}(\overline{S}_i) = \overline{G}$ then $S = \overline{S}_i$ is simple RG-module with vertex P.

2. If $Stab_{(\overline{G})}(\overline{S}_i) = H_{p'}\overline{G}$ then $U = \overline{S}_i \uparrow^{\overline{G}}_{H_{p'}(\overline{G})}$ is semisimple and each direct summand S is a simple projective $R\overline{G}$-module with vertex $O_p(G)$.

Observe that $k\uparrow_P^G \cong k[H_{p'}(G)]$ as kG-modules. The kG-module $k[H_{p'}(G)]$ can be decomposed as a direct sum of simple kG-modules, since $Rad(k[H_{p'}(G)]) = 0$. From $k\uparrow_P^G\uparrow^G_{H_{p'}(G)} \cong kG$ it follows that any simple kG-module is the direct sum of simple $k[H_{p'}(G)]-$modules. Further more these t of this direct sum m and s contains a complete list of simple RG-modules. In fact, if the set $x_1,...,x_{|P|}(x_i \in P)$ is a set of representatives of the left cosets $G/H_{p'}(G)$ then any element $g \in G$ can be expressed in the form $g = xy$ where $x \in P$ and $y \in H_{p'}(G)$. Thus we make use of the following general isomorphism

$$kG \cong kP \otimes_k k[H_{p'}(G)] \tag{3.0.2}$$

which arises because kG has as a basis the elements xy where $x \in P$ and $y \in H_{p'}(G)$, and $kP \otimes_k$ k(Hp´(G)(has as a basis the corresponding elements x (y.

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1. By assumption $S = \overline{S}_i$ is simple. Since $k\uparrow_P^G \cong k[H_{p'}(G)]$ the result follows.

2. In this case we have $Rad(\overline{S}_i) = Rad(kG)\overline{S}_i = \bigoplus_{g\in[\overline{G}/H_{p'}(\overline{G})]} Rad(g \otimes \overline{S}_i)$, where the $g \otimes \overline{S}_i$ are simple $k[{}^gH_{p'}(\overline{G})]$-modules, so Rad(U) = 0. Since $\overline{S}_i$ is projective $k[Hp'(\overline{G})]$-module U is projective $k\overline{G}$-module. Hence each direct summand S is also direct summand of $k\uparrow^{\overline{G}}_{O_p(G)} \cong k\overline{G}$. Thus there sult follows. □

Let R be a complete discrete valuation ring with maximal ideal (Π) and residue field $k = R/(\Pi)$. Let G be a finite group with p-complement and splitting field k. Assume that S is a simple RG-module. Then the finitely-generated RG-module Q is called quasi-simple RG-module if it is free as R-module and $Q/(\Pi)Q \cong S$ for some simple RG-module S.

Theorem 3.0.6. Let G be a finite group with p-complement $H_{p'}(G)$. Every simple RG-module S has a quasi-simple RG-module.

Proof. Let $k[H_{p'}(G)] = \hat{S}_1 \oplus ... \oplus \hat{S}_m$ be the decomposition of $k[H_{p'}(G)]$ as direct sum of simple $k[H_{p'}(G)]$-modules. Then we may write $R \otimes_R k[H_{p'}(G)] \cong R[H_{p'}(G)] = R \otimes_R \hat{S}_1 \oplus ... \oplus R \oplus_R \hat{S}_m = P_{\overline{S}_m} \otimes ... \otimes P_{\overline{S}_m}$ where $P_{\overline{S}_i}$ is the projective cover of S_i. If $Stab_G(S_i) = G$ then $Q_{S_i} = P_{S_i}$ is a quasi-simple RG-module.

Let $U = S_1 \otimes ... \otimes S_t$ be a decomposition of U as direct sum of simple RG-modules. Then $R \otimes_R U = RU = R \otimes_R S_1 \otimes$ … (R(R Sn = QSi Q_{S_1} … Q_{S_t} = where each RG-module QSi is indecomposable R[Hp´(G)]-module and free as R-module. Hence QSi /(Π) $Q_{S_i} \cong k \otimes S_i$ which complete the proof. (

The quasi-simple RG-module Q_S is called quasi-simple RG-module corresponding to S. Let R be a discrete valuation ring with maximal ideal (□) and residue field k = R/(P) of characteristic p. Let $\varphi : G \to GL(n, R)$ be a representation of G. Then the homomorphism $\overline{\varphi} : G \to GL(n, k)$ induced by the natural projection $GL(n, R) \to GL(n, k)$ is called reduction of φ modulo p.

Lemma 3.0.7. Let G be a finite group and let Q be a quasi-simple RG-module. Assume that $\varphi : G \to GL_n(Q)$ is a representation of G and $\overline{\varphi} : G \to GL_n(Q/\Pi)Q)$ the reduction of φ modulo p. Set $L = \langle \varphi_g \mid g \in G\rangle$ and $\overline{L} = \langle \overline{\varphi}_g \mid g \in G\rangle$. Then rank $\overline{L} \le rank_R L$.

Proof. Set m = $rak_R L$. Let $d_1,\ldots,d_m$ be a R-basis of L. Consider the surjection $\sigma: L \to \overline{L}$ given by $\sigma(d_i) = \overline{d}_i$ (i = 1,…,m). Thus $\overline{g}_1,\ldots,\overline{g}_m$ is spanning set for $\overline{L}$. Hence the result follows. □

Theorem 3.0.8. Let G be a finite group. Let φ : □G → □$GL_n(Q)$: be an R-representation of G. Then Q is a quasi-simple RG-module if and only if $\langle \varphi_g \mid g \in G\rangle = End_R(Q) \cong M_n(R)$, where $M_n(R)$ denotes the ring of the $n \times n$-matrices over R.

Proof. If Q is quasi-simple then $\overline{\varphi}$: G → □$GL_n(Q/\Pi)Q)$ is absolutely simple, since k is a splitting field of G. Therefore $\langle \overline{\varphi}_g \mid g \in G\rangle = End_R(Q/\Pi)Q)$ by the classical Burnside theorem. According to the last lemma it follows that $rank_R\langle \varphi_g \mid g \in G\rangle = n^2$. Conversely, if $\langle \varphi_g \mid g \in G\rangle = = End_R(Q)$ then $\langle \overline{\varphi}_g \mid g \in G\rangle = End_R(Q/(\Pi)Q)$. Hence $End_k(Q(\Pi)Q)^G = (End_{kG}(Q/(\Pi)Q) = Z(End_k(Q/(\Pi)Q)) \cong k$, so $Q(\Pi)Q$ is absolutely simple. This completes the proof of this implication. □

Theorem 3.0.9. Let G be a finite group. Then the RG-module Q is a quasi-simple RG-module if and only if $End_{RG}(Q) \cong R$, with the endomorphisms acting as scalar multiplication.

Proof. Let φ be an R-representation of G corresponding to Q. According to the last theorem we have $\varphi_g \mid g \in G\rangle = End_R(Q)$. Since $End_R(Q)^G = End_{RG}(Q)$ it follows that $Z(End_R(Q)) = End_{RG}(Q)$. Conversely if $End_{RG}(Q) \cong R$ then $End_{kG}(Q/(\Pi)Q) \cong k$. Hence $Q/(\Pi)Q$ is absolutely simple RG-module, which complete the proof of this of this implication. □

Lemma 3.0.10. Let $G = K \times H$ where K is a p-group and H has order prime to p. If S is any simple RG-module then $\hat{P}_S = RK \otimes_R Q_S$.

Proof. Since kH is semisimple we may write $kH = k \oplus U$ for some kH-module U. Thus $\hat{P}_R = R$ is a projective RH-module and we may write $RH = R \oplus \hat{U}$ for some projective RH-module $\hat{U}$, and now $RG = RH \uparrow_H^G \oplus U \uparrow_H^G$. Here $R \uparrow_H^G \cong RK$ as RG-modules, and so RK is projective, being a summand of RG. Therefore $RK \otimes_R R\, Q_S$ is projective (See [3] proposition 8.4). Now

$$RK \otimes_R Q_S/(\Pi)(RK \otimes_R Q_S) \cong kK \otimes_R S.$$

We have

$$Rad(kK \otimes_R S) \supseteq (Rad(kK)kK) \otimes_R S.$$

since K is a normal p-subgroup and so acts trivially on S. As $Rad(kK)kK) \otimes_k S$ has codimension in $kK \otimes_R S$ equal to dim S we have that $kK \otimes_k S/Rad(kK \otimes_k S) \cong S$. Since $kK \otimes_k S$ is projective we obtain that

$$RK \otimes_R Q_S/Rad(RK \otimes_R Q_S) \cong S.$$

By proposition (2.0.3) we conclude that $\hat{P}_S = RK \otimes_R Q_S$.

Theorem 3.0.11. Let G be a finite group containing a subgroup $\tilde{G}$. Fixed $P \in Syl_p(G)$. Then

$$\hat{P}_S = \begin{cases} RP \otimes_R Q_S & \text{if} \quad p \mid \dim S \\ RO_p(G) \otimes_R Q_S & \text{otherwise} \end{cases}$$

Proof. Let $\tilde{S}_1, \ldots, \tilde{S}_n$ be a complete list of non-isomorphic simple $R\tilde{S}$ – modules. According to the last lemma we may write

$$R\tilde{G} = RO_p(G) \otimes_R Q_{\tilde{S}_1} \otimes \ldots \otimes RO_p(G) \otimes_R Q_{\tilde{S}_n} .$$

Now

$$RG = R\tilde{G} \uparrow_{\tilde{G}}^{G} = (RO_p(G) \otimes_R Q_{\tilde{S}_i}) \uparrow_{\tilde{G}}^{G} \oplus \ldots \oplus (RO_p(G) \otimes_R Q_{\tilde{S}_n}) \uparrow_{\tilde{G}}^{G} .$$

Notice that

$$\begin{aligned}(RO_p(G) \otimes_R Q_R) \uparrow_{\tilde{G}}^{G} &= (RO_p(G) \otimes_R R) \uparrow_{\tilde{G}}^{G} \\ &\cong (RO_p(G) \uparrow_{\tilde{G}}^{G} \\ &\cong RP.\end{aligned}$$

Thus RP is projective, being a direct summand of RG. The set $x_1, \ldots, x_{|P|}$ with $x_i \in P$ is a set of representatives of the left cosets $G / H_{p'}(G)$. Thus any element $g \in G$ can be expressed in the form $g = xy$ where $x \in P$ and $y \in Hp'(G)$. We make use of the following general isomorphism

$$RG \cong RP \oplus_R R[H_{p'}(G)]$$

which arises because kG has as a basis the elements xy where $x \in P$ and $y \in H_{p'}(G)$, and $RP \oplus_R R[H_{p'}(G)]$ has as a basis the corresponding elements $x \otimes y$. Let $R[H_{p'}(G)] = Q_{\tilde{S}_1} \oplus \ldots \oplus Q_{\tilde{S}_n}$ be the decomposition of $R[H_{p'}(G)]$ as direct sum of quasisimple $R[H_{p'}(G)]$-modules. Then

$$RG \cong RP \oplus_R Q_{S_1} \oplus \ldots \oplus RP \otimes_R Q_{S_n},$$

We have to check two cases.

- $\mathrm{Stab}_G(\tilde{S}_i) = G$. In this case $S = \tilde{S}_i$ is a simple RG-module and $p \nmid \dim S$. Now

$$\mathrm{Rad}(RP \otimes_R Q_S) \supseteq \mathrm{Rad}(RP) \otimes_R \mathrm{Rad}(Q_S).$$

Therefore

$$\begin{aligned}RP \otimes_R Q_S / \mathrm{Rad}(RP) \otimes_R \mathrm{Rad}(Q_S) &= RP / \mathrm{Rad}(RP) \otimes_R Q_S / \mathrm{Rad}(Q_S) \\ &\cong k \otimes_k S \\ &\cong S\end{aligned}$$

Since R is a complete discrete valuation ring, $RP \otimes_R Q_S / \mathrm{Rad}(RP \otimes_R Q_S)$ is semisimple kG-module it follows that

$$RP \otimes_R Q_S / \mathrm{Rad}(RP \otimes_R Q_S) \cong S.$$

This shows that $RP \otimes_R Q_S$ is projective cover of S.

• $\mathrm{Stab}_G(\tilde{S}_i) = \tilde{G}$. In this case we have

$$\begin{aligned}(RP \otimes_R Q_{\tilde{S}_i}) \uparrow_{\tilde{G}}^{G} &= RO_p(G) \uparrow_{\tilde{G}}^{G} \otimes_R Q_{S_i} \\ &\cong RO_p(G) \otimes_R Q_{\tilde{S}_i} \uparrow_{\tilde{G}}^{G} \\ &\cong RO_p(G) \otimes_R (S_1 \otimes \ldots \otimes S_t) \\ &\cong RO_p(G) \otimes_R S_1 \otimes \ldots \otimes ROp(G) \otimes_R S_t)\end{aligned}$$

Thus $RO_p(G) \otimes_R Q_{S_j}$ is projective. We may now proceed as in the previous case. □

4. ALPERIN'S WEIGHT CONJECTURE

Let G be a finite group and k an algebraically closed field of characteristic p. Alperin defines a weight to be a pair (P, V) where $P \leq G$ is a p-subgroup and V is a p-block of defect zero of $N_G(P) / P$. We take P up to conjugacy and V up to isomorphism. **Alperin Weight Conjeture**. The number of weights of G equals the number of conjugacy classes of elements of G of order prime to p (AWC). The conjecture connects two sets which otherwise are seemingly unrelated. The simple modules are enumerated by Brauer's theorem and this is easy. The blocks of defect zero are notoriously dificult to pin down in generality. The conjecture is known to be true for

1. p-solvable groups (Okuyama)

2. Groups of Lie type in defining characteristic.

3. Symmetric groups

4. Many sporadic simple groups

Lemma 4.0.12. Let G be a finite group with p-complement $H_{p'}(G)$. The Green correspondence induces a bijection between the simple kG-modules with vertex $P \in \mathrm{Syl}_p(G)$ and the non-isomorphic simple $k[N_G(P)]$-modules.

Proof. Let S be a simple kG-module with vertex P. Then $S\uparrow_{N_G(P)}^{G} = f(S) \oplus X$ where f(S) is the unique indecomposable summand with vertex P in any decomposition of $S\uparrow_{N_G(P)}^{G}$ as direct sum of indecomposable modules and each summand of X is projective relative to a subgroup of the form $N_G(P) \cap {}^g P$ with $g \in G - N_G(P)$ by Green correspondence theorem part 1(See [3] theorem 11:26). Since P is a Sylow p-subgroup $P = O_p(N_G(P))$, i.e., P is a p-radical subgroup. Hence we can write $N_G(P) = P \times H$, where $|N_G(P) : H| = |P|$, by the classical Shur-Zassenhaus theorem. For any element $g \in G$ can be expressed as a product xy with $x \in P$ and $y \in H_{p'}(G)$, so $N_G(P)$ contains a p-complement $H_{p'}(N_G(P)) \leq H_{p'}(G)$. Since S is a simple $k[H_{p'}(G)]$-module and $H_{p'}(NG(P))$ has order prime to p we can write $S\uparrow_{H_{p'}(N_G(P))}^{G} = V_1 \oplus \ldots \oplus V_m$,where the V_i are simple $k[H_{p'}(NG(P))]$-modules whit vertex P. Since the simple $k[N_G(P)]$-modules are precisely the simple $k[H_{p'}(NG(P))]$-modules we deduce that f(S) is a simple $k[N_G(P)]$-module and X = 0. If V is a simple $k[N_G(P)]$-module then $V\uparrow_{N_G(P)}^{G} = g(V) \oplus Y$, where g(V) is an indecomposable kG-module with vertex P and each summand of Y is an indecomposable projective relative to a subgroup of the form ${}^gP \cap P$ where $g \in G - N_G(P)$ by Green Correspondence part 2. Now $kP \otimes V$ is an indecomposable projective $k[N_G(P)]$-module by theorem (3:0:11). Therefore $kP \otimes V\uparrow_{N_G(P)}^{G}$ is a projective kG-module, so $kP \otimes g(V)$ is also projective kG-module and we have $kP \otimes g(V) = P_{S_1} \oplus \ldots \oplus P_{S_n}$ for various indecomposable projective modules P_{S_i}. Thus it follows that $kP \oplus g(V) / \mathrm{Rad}(kP \oplus g(V)) \cong k - g(V) \cong g(V) \cong S_1 \oplus \ldots \oplus S_n$ by uniqueness of projective covers. Since g(V) is indecomposable kG-module we obtain n = 1 and $g(V) \cong S_1$. By Green correspondence theorem par 3 it follows that $gf(S) \cong S$ and $fg(V) \cong V$. So we are done. □

Theorem 4.0.13. Let G be a finite group with p-complement $H_{p'}(G)$ and let k be a field of characteristic p. Then G satisfy (AWC).

Proof. Let $S_1,\dots,S_n$ a complete list of simple kG-modules with vertex $P \in Syl_p(G)$. According to the last lemma there is a complete list $V_1,\dots,V_n$ of simple $k[N_G(P)]$-modules. Since P is p-radical subgroup we have $N_G(P) = P \times H_{p'}(N_G(P))$. Hence the simple $k[N_G(P)]$-modules are precisely the simple $k[H_{p'}(N_G(P))]$-modules. Therefore the pairs (P, V_i) are weights for G. Let $U_1,\dots,U_m$ be a complete list of simple kG-modules with vertex $O_p(G)$. Applying again the lemma (4:0:12) we obtain that the pairs $(O_p(G); U_i)$ are weights for G. Suppose that the pair (D; M) is a weight for G. Assume that $O_p(G) < D < P$. Since g = xy with $x \in P$ and $\in 2\ H_{p'}(G)$ for all $g \in G$ we deduce that $N_G(D)$ contains a p-complement $H_{p'}(N_G(D)) \cdot H_{p'}(G)$. By Green correspondence theorem we have $M\uparrow^{H}_{N_G(D)} = g(M) \oplus Y$ where g(M) is an indecomposable kG-module with vertex D. Since $kD \otimes M$ is an indecomposable projective $N_G(D)$-module $(kD \otimes M)\uparrow^{H}_{N_G(D)}$ is projective kG-module, so $(kD \otimes g(M))$ is also projective kG-module. Thus $kP \otimes g(M) = P_{S_1} \oplus\dots\oplus P_{S_n}$ for various indecomposable projective modules P_{S_i}. Since g(M) is indecomposable we deduce that $g(M) \cong S_1$, a contradiction. So we finish the proof. □

5. BLOCKS OF FINITE GROUPS WITH P-COMPLEMENT

Lemma 5.0.14. Let G be a finite group with p-complement $H^{p'}(G)$, and let (F; R; k) be a p-modular system.

1. FP and $FO_p(G)$ can be expressed as direct sum of simple FG-module.

2. $F \otimes_R Q_S$ is a simple FG-module.

Proof.

1. Since $F\uparrow^{G}_{H} \cong FP$ as FG-module and $FO_p(G)$ is an FG-module with the action of G given by conjugation the result follows.

2. By theorem 3:0:9 we have $End_{RG}(Q_S) = R$, so $End_{RG}(F \otimes_R Q_S) = F$. The result follows by the classical Schur's lemma. □

Assume that $D = P \in Syl_p(G)$ or $D = O_p(G)$.

Theorem 5.0.15. Let G be a finite group with p-complement $H_{p'}(G)$, and let (F; R; k) be a p-modular system. Then every simple kG-module is the reduction module (Π) of an RG-lattice.

Proof. We have $\hat{P}_S \cong RD \otimes_R Q_S$.Therefore

$$F \otimes_R \hat{P}_S = F \otimes_R (RD(G) \otimes_R Q_S)$$

$$= (F \otimes_R RD(G)) \otimes_R Q_S.$$

According to the last lemma we have

$$F \otimes_R RD(G) \cong FD \cong F \oplus U$$

for some FG-module U. Hence

$$(F \otimes_R RD(G)) \otimes_R Q_S \cong (F \otimes_R Q_S) \oplus (U \otimes_R Q_S).$$

We may now apply again the last lemma. (

Theorem 5.0.16. Let G be a finite group with p-complement Hp´(G). Each block of Kg can be induced from Hp´(G) to G.

Proof. Observe that if $1 = e_1 + \ldots + e_m$ is the sum of blocks $k[O_{p'}(G)]$ then for each i and $g \in G$ the conjugate ge_ig^{-1} is also a block of $k[O_p(G)]$. For this we verify that this element is idempotent, and also that it is central in $k[O_{p'}(G)]$, which is so some if $x \in O_{p'}0(G)$ then $x(ge_ig^{-1}) = g(g^{-1}xg)e_ig^{-1} = (ge_ig^{-1})x$. Furthermore ge_ig^{-1} is primitive in $Z(kO_{p'})$ since if it were the sum of two orthogonal central idempotents, on conjugating back by g^{-1} we wold be able to deduce that ei is not primitive either. Since the blocks of $k[O_{p'}(G)]$ are uniquely determined, and it follows that $ge_ig^{-1} = e_j$ for some j. Thus G permutes the blocks of $k[O_{p'}(G)]$. We denote the G-orbit of e_i by $Orb_G(e_i)$. We define $f = \sum_{j=1}^{|Orb_G(e_i)|} (e_i)$, where e_j ($Orb_G(e_i)$. Clearly f is idempotent. It is in fact central in k $[O_{p'}(G)]$ since if $g \in G$ then $gfg^{-1} = \sum_{j=1}^{|Orb_G(e_i)|} ge_jg^{-1} = f$, the sum again being over the elements in $Orb_G(e_i)$. We now show that f is primitive in Z(kG). Suppose instead that $f = f_1 + f_2$ is a sum of orthogonal idempotents in Z(kG). Then $gf_ig^{-1} = f_i$ for any $g \in G$. Since f is the sum of the elements of the G-obit of e_j we deduce that $f_1 = f_2$, a contradiction. This shows that f is primitive in Z(kG). Thus we can write $k[O_{p'}(G)] = A_1 \otimes \ldots \otimes (A_m$ where the A_i are 2-sided ideals of kG. Since $k[O_{p'}(G)]\uparrow^{G}_{O_{p'}} \cong kG$ we obtain the sum

$$1 = d_1 + \ldots + d_n \tag{5.0.3}$$

where each summand di is a block of kG. So we are done.

Lemma 5.0.17. Let G be a finite group with p-complement $H_{p'}(G)$. Then $C_G(O_{p'}(G))$ contains a p-complement.

Proof. Obvious.

Theorem 5.0.18. Let G be a finite group with p-complement $H_{p'}(G)$. If $O_{p'}(G)$ is the p-complement of $C_G(O_p(G))$ then each block of kG lies in $k[O_{p'}(G)]$, and is the sum of a G-conjugacy class of blocks of $k[O_{p'}(G)]$.

Proof. Since $O_p(G)$ is a normal p-subgroup every block of kG lies in $k[CG(O_{p'}(G))]$ and is the sum of a G-orbit of blocks of $C_G(O_p(G))$(See [3] proposition 15:9). Furthermore each block of $k[C_G(O_p(G))]$ is the sum of a $C_G(O_p(G))$-conjugacy class of blocks of $k[O_{p'}(G)]$ (See [3] proposition 12:7). Since each block of kG is stable under conjugation by G the result follows.

We denote the fixed point set of $O_{p'}(G)$ under conjugation in G by $J_{p'}(G)$. We may check that $J_{p'}(G)$ is a normal p-subgroup of G, which is immediate.

Theorem 5.0.19. Let G be a finite group with p-complement $H_{p'}(G)$ and $P \in Syl_p(G)$. Then $O_{p'}(N_G(P)) = J_{p'}(G)$.

Proof. Since P is radical subgroup we may write $N_G(P) = P \times H_{p'}(N_G(P))$ by the classical Schur-Zassenhaus theorem, where $H_{p'}(N_G(P))$ is a p-complement of $N_G(P)$. As $J_{p'}(G) \leq C_G(P)$ is the p-complement of $C_G(P)$ and $O_{p'}(N_G(P)) \cdot C_G(P)$ the assertion follows. □

Definition 5.0.20. If (P, V) is a weight for G, then V belongs to a block b of $N_G(P)$ and this block corresponds with a block Bl of G via the Brauer correspondence, hence we can say that the weight (P, V) belongs to the block Bl of G so the weights are partitioned into blocks.

(**Alperin's Conjeture Block Form**) The number of weights in a block of G equals the number of simple modules in the block(ACBF).

Lemma 5.0.21. Let G be a finite group with p-complement $H_{p'}(G)$ and S and T two simple modules with vertex $P \in Syl_p(G)$. Assume that Bl is a block of G and this block corresponds with a block b of $N_G(P)$ via the Brauer correspondence. If S and T lie in the block Bl with defect group P then $S \downarrow^{G}_{N_G(P)}$ and $T\downarrow^{G}_{N_G(P)}$ lie in the block b with defect group P.

Proof. If S and T lie in the same block then there is a list of simple kG-modules $S = S_1,\ldots,S_n = T$ with vertex P such that S_i and S_{i+1} are both composition factors of the some indecomposable projective module $kP \otimes S_i$ for each i = 1,…,n - 1. By lemma (4.0.12) we have a list of simple $k[N_G(P)]$-modules $S\downarrow^{G}_{N_G(P)} = V_1,\ldots,V_n = T\downarrow^{G}_{N_G(P)}$.

Suppose that $0 \subset P_1 \subset \ldots P_n$ is a composition series of the regular representation of P. Since P is a p-group, all the composition factors are trivial representation, k. Because - $\otimes_k S_i$ preserves exact sequences we have $0 \subset P_1 \otimes_k Si \subset \ldots P_n \otimes_k S_i = P_{S_i}$. Therefore V_i and V_{i+1} are both composition factors of the some indecomposable projective module $kP \otimes_k V_i$ for each i = 1,…,n - 1. Thus the result follows. □

Theorem 5.0.22. Let G be a finite group with p-complement $H_{p'}(G)$ and let k be a field of characteristic p. Then G satisfy (ACBF).

Proof. If (P, V) and (P, U) belongs to the block Bl then g(V) and g(U) belongs to the block Bl by last lemma. If $(O_p(G); S)$ belongs to the same block Bl then S lie in the block Bl, since $N_G(O_p(G)) = G$. So we are done. □

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