NETWORK DESIGN APPLICATIONS OF THE CLASS OF COLUMN GENERATION/SIMPLICIAL DECOMPOSITION ALGORITHMS IN CONVEX DIFFERENTIABLE OPTIMIZATION

Ricardo García Ródenas, Universidad de Castilla La Mancha[1]
Angel Marín Gracia, Universidad Politécnica Madrid[2]
Michael Patriksson, Chalmers University of Technology, Gothenburg[3]

ABSTRACT
A new class of column generation/simplicial decomposition method for non linear convex and differentiable programming is presented. The new algorithm class builds on the intuitively appealing idea that non linear column generation problems may be advantageous computationally. Difference applications of this methodology are presented, with special attention to the unicommodity and multicommodity network flow problems, which are obtained when some decomposition methods applied to network design problems.

Key words: convex and differentiable programming, column generation, network design, capacitated network design, multicommodity flow problems, design of urban multimodal interchanges, and rail freight network design.

MSC: 90C35

RESUMEN
Una nueva clase de métodos de descomposición generación/simplicial de columna para la programación no lineal convexa y diferenciable es presentada. La nueva clase de algoritmos se construye sobre la idea intuitiva de que los problemas de generación de columna pueden ser ventajosos desde el punto de vista computacional. Diferentes aplicaciones de esta metodología son presentadas, poniendo atención especial a problemas del tipo "unicommodity" y "multicommodity" para el flujo de redes, los cuales son obtenidos cuando algunos métodos de descomposición son aplicados a problemas del diseño de redes.

Palabras clave: programación lineal convexa y diferenciable, generación de columna, plan de la red, plan de la red capacitado, problemas de flujo en multicomodidad, intercambio en el plan multimodal urbano, plan de red de carga ferroviario.

1. INTRODUCTION

The Network Design Problem (NDP) arises in a variety of problems for solving the balance between investment in facilities and the operative cost to use the network. The NDP is defined on a graph, where some resources of facilities can be located in the nodes and arcs of the graph, which give the capacity of them. The demand origin-destination is distributed through the available routes of the network.

The cost to minimize is defined adding the cost for resources assignment and the cost for routing the commodities through the network. The NDP constraints are the specific routing and resource ones and the relation between them; the commodity route flows must be inferior to the arc and/or node capacity given by resources assignment.

The Fixed-Charge Facility Location Problem establishes the capacity at the arcs of multicommodity flow network problems. Gendron and Crainic (1994) propose different relaxation together with heuristics for yielding feasible solution. Hermann **et al**. (1996) study the capacitated network design problem (CNDP) using a dual ascent approaches. Holmberg and Yuan (1996) use an efficient method based on Lagrangian heuristic within a branch and bound framework.

E-mail:[1]rgarcia@pol-al.uclm.es
[2]amarin@dmae.upm.es
[3]mipat@mathchalmers.se

NDP have wide applications in telecommunication and transportation network design. See the surveys of Magnanti and Wong (1984), Minoux (1989); Chapter 16 of the book by Ahuja **et al**. (1993); Chapter 1, Section 13 of Ahuja **et al**. (1995).

More specifically, for telecommunication network design, Balakrishnan **et al**. (1991) and Chapter 1 of the book by Gendron **et al**. (1998) present important review in the subject.

Parris **et al.** (1992) combine network topology (node and arc location) and capacity expansion problems for leased telecommunication network, in which there is a hierarchy of possible transmission facilities. They propose a double heuristic: a lower level a single-period topology only with operative costs, and at upper level with all the costs.

Chang and Gavish (1993) formulate a network topology and capacity expansion model in a telecommunication network (multicommodity, node location given and link capacity expansion). They include schemes exploiting the underlying special structure by Lagrangian relaxation and a global search strategy.

For transportation applications several application may be mentioned, here two of them have been chosen given our direct experience. The Rail Freight Network Design (RFNDP) and the Urban Multimodal Interchange Design (UMIDP) Problems.

The RFNDP considers the design of the optimal frequency and the schedule of the trains, which define the line services, through then *commodities* of the demand are routing. At the stations the car *classification* is produced using different parallel tracks. The cars are sorted in such a way that each track contains cars with a common (possibly intermediate) destination. For tactical planning level see, Crainic **et al**. (1984), Crainic **et al**. (1986), Keaton (1992) and Marín and Salmerón (1996).

For urban transportation it may be mentioned the UMIDP, where it must be decided the three following aspects: location of the interchanges in the main transit network, the design of facilities in these interchanges, and the design of the access mode to the interchanges.

Oppenheim (1994) describes the network design process as a bi-level programming problem, the upper level problem being the design problem and the lower level problem being the demand problem of these services. The system designer (leader) designs the system of transport taking into account how the users (followers) employ it. The leader is assumed to have knowledge of the responses of the followers. This situation is known in game theory as a *Stackelberg game.*

The bi-level programming is difficult to solve in practice due to its inherent nonconvexity and nonsmoothness. Friesz **et al**. (1992) use Simulated Annealing to solve the bi-level programming formulation of the continuous network design problem.

García and Marín (1999) consider simultaneously aspects of continuous and discrete network design, such as: capacity and fares for the first one and location of facilities and design of the secondary network for the second one. They have combined the simulating annealing method and heuristic for solving it.

NDP are frequently decomposed into convex and differentiable subproblems. Each one must be solved at each iteration of the decomposition scheme, so it is very important, improve the efficiency of the methods used to solve them.

So pseudo-convex minimization problems arise in a large variety of applications in the engineering and decision making. These applications require of the decomposition methodology for solving very large scale problems. In this framework the following constrained differentiable optimization problem is considered:

$$CDP(f;X): \underset{x \in X}{Min} f(x),$$

where f is pseudo-convex and continuous differentiable function and X is a non empty, convex and compact set.

García **et al**. (1999a) present a class of algorithms for these large-scale nonlinear optimization problems. There are two main characteristics of the methods in this class: (i) an approximation of the original problem in constructed and solved, where in the original feasible set is replaced by a convex and compact subset, which is an inner approximation, and it is called the restricted master problem (RMP); and (ii) this inner approximation is improved (that is, enlarged) by generating a column set in the feasible set through the solution of another approximation of the original problem, column generation problem (CGP) where in the original objective function is approximated.

As such, this class of column generation/simplicial decomposition (CG/SD) methods may be placed within the framework of Column Generation methods. Another characteristic of a CG/SD method is that the sequence of solution to the inner approximated problems tends to a solution to the original problem is such way that the objective function is a strictly monotone approach at its optimal value. Therefore, the class of CG/SD methods also falls within the framework of iterative descent algorithms.

This modular algorithm can be formulated in general as:

- Algorithm to Solve Column Generation Problem (CGP). Let $x \in X$, so solve the following problem:

$$CGP(\Pi; X) : \underset{y \in X}{Min}(y; x)$$

where Π is a merit function. This approximation problem is much easier to solve than the original one. The structures of the set X and of the function Π are used in order to solve it efficiently. Note that the original function f can be used as a merit function, that is Π = f.

- Algorithm to Solve Restricted Master Problem (RMP). Let $\overline{X}$ be a convex and compact subset of X such that [x,y] is a subset of $\overline{X}$, where y is a solution of CGP(Π,X). Solve the following optimization problem:

$$RMP(f; \overline{X}) : \underset{x \in \overline{X}}{Min} f(x)$$

The RMP have two main characteristics: (i) at first iterations it has less number of variables, (ii) it has a very simply feasible region, normallly a simplex. These characteristics may be able to use nonlinear codes that have the important feature to have superlinear convergence.

2. COLUMN GENERATION STATE OF ART

The classical form of a CG/SD method is Simplicial Decomposition (SD), originating in the works of Holloway (1974) and von Hohenbalken (1977). This algorithm was described for problems where X is a polyhedral set.

A profitable extreme point or direciton of X is generated through the solution of CGP(Π;X), in which Π is a first-order linear approximation of f, i.e., $\Pi(y,x) = f(x) + \nabla f(x)^T(y-x)$, defined at the solution, x, of $RMP(f, \overline{X})$. This approximate problem is a linear programming problem, which in general is much easier to solve than the original one.

Let knowing the subset $\overline{P}$ a subset of extreme points of the compact polyhedral set X. The inner approximation $\overline{X}$ of X is defined by the (polyhedral) set of points that can be expressed as a convex combination of these extreme points. The advantage of $RMP(f, \overline{X})$ respect to original problem in that the inner representation of X is much simpler than its original, outer, representation in terms of linear equalities and inequalities; disregarding the definition constraints. The set $\overline{X}$ is described by the Cartesian product of a simplex and the non-negative orthant, an optimization over which often can be made with little more effort than for an unconstrained problem, see Bertsekas (1982).

The disadvantage of SD is that since the number of extreme points and directions of a polyhedral set grows exponentially with its dimension, and the $RMP(f, \overline{X})$ introduces an impractical large number of variables when the SD progress.

For such disadvantage, an improvement over the original scheme referred to as restricted simplicial decomposition (RSD), was devised by Hearn **et al**. (1985 and 1987). They devise a modification of the original scheme, in which the number of extreme points retained is kept below a positive integer, r. When this number of extreme points has been reached, any new extreme point generated replaces the column in $\overline{P}$ that received the least weight in the solution to the RMP. In order to ensure the convergence of the algorithm, the optimal solution x to the RMP must also be retained as an individual column (however not counted among the r columns). This algorithms is again an instance of the class CG/SD where $\overline{X}$ is defined by means of the previous rule.

An extension of the RSD algorithm was made in Larsson **et al**. (1966 and 1997). The motivation behind the non-linear simplicial decomposition (NSD) method is that by generating high quality columns based on better approximations of the objective function in CGP, a less amount of columns will be needed to describe an optimal solution, and the sensitive of the method to the dimension of the optimal face will be reduced, moreover the number of main iterations is small when small values of the parameter r are chosen. Further, more efficient methods can be applied to each RMP, since they are smaller.

The NSD method is obtained from the CG/SD method by choosing the objective function of CGP:

$$\Pi_{NSD}(y;x) = f(x) + \nabla f(x)^T(y-x) + \varphi(y;x)$$

where $\varphi(y;x)$ is a continuous function, convex and continuously differentiable respect to y for all x belongs to X, and with the properties that: $\varphi(x;x) = 0$ and $\nabla_y \varphi(x;x) = 0$ for all x belongs to X.

García **et al**. (1999a) generalize both CGP and RMP in NSD algorithms. First, they consider a more general class of objective functions in CGP, that are merit functions, Π, and by other side, they take into account more general feasible set in RMP.

The convergence results for CG/SD methods allow both CGP and RMP are inexactly solved, thus facilitating its practical use. This is a crucial result that opens a way to see the CG/SD methods, if we choose the original function as merit function the CGP is defined by a crude algorithm used for solving it. This choice changes the vision of coordination of subproblems by coordination of descent closed algorithms.

3. CG/SP PROPERTIES

We begin by starting the CG/SD conceptual scheme, which allows for the truncated solutions of both the CGP and RMP, and further it allows for very general rules for updating the inner approximations. The algorithm is described by means of (possibly point-to-set) closed descent algorithmic mapping, which will be assumed to fulfill conditions similar to those utilized in the convergence analysis in Zangwill (1969).

The CGP is solved using an iterative procedure, denoted by A_r^k and belonging to a finite collection k_c of two possible types. In the first alternative, the mapping is applied once only in a main iteration, and then it is assumed to provide a descent direction unless a solution is at hand; we denote this by type 1. The second alternative is based on the existence of a merit function, Π. In this case, the algorithm is applied at least once and possibly any number of times, starting from the point x^k at which the restricted master problem was terminated, and then it is assumed to yield a reduced value of the merit function Π each time it is applied, unless a solution is at hand; we denote these ones by type 2.

The RMP is assumed to be solved by an iterative procedure, denoted A_c^k and belonging to a finite collection K_r with similar properties to the above second case, with the exception that the algorithm should reduce the value of f each time that it is applied unless a solution to the CDP is at hand.

In Table 1 the CD/SD algorithm is defined

Table 1. The CG/SD algorithm.

0. (Initialitation): choose and initial point x^0, and let t: = 0.
1. (Column generating problem): choose an algorithm A_c^k of the set K_c. It it of type 1, then apply one iteration, otherwise at least one iteration of the algorithm, on CDP(f;X), starting from x^t. Let the resulting point be y^t.
2. (Termination criterion): if x^t solves CDP(f;X) stop. Otherwise, continue.

3. (Set augmentation): let x^{t+1} a subset of X be a non-empty, compact and convex set such that the line $[x^t, y^t]$ belongs to X^{t+1}.

4. (Restricted master problem): choose an algorithm A_r^k of the set K_r. Apply at least one iteration of the algorithm on CDP(f;X^{t+1}), starting from x^t. Let the resulting point be x^{t+1}.

5. (Update): Let t: = t+1. Go to Step 1.

Global convergence results are given for pseudo-convex minimization over compact and convex feasible sets. García **et al**. (1999a). Patriksson (1999) is a good reference about the theory used.

Hearn **et al**. (1985) proved that X^t is a simplex for all t in RSD and SD under the hypothesis that RMPs are solved exactly and using some given rules. García **et al**. (1999a) have shown that this property is keep in CG/SD algorithms. The finitness convergence, in general, is lost for these algorithms. These authors give sufficient conditions to guaranty the finite convergence properties (identifying the optimal face or an optimal solution), and extended to the solution of variational inequality problems. These properties depend on the geometry of the problem, the properties of the algorithm used in CGP and the construction and solution of RMP. We have used the following hypothesis:

1. SOLF(f;X) is a set of weak sharp minima for CDP(f;X)
2. The algorithm used in CGP forces to zero the *projection of the gradient* into X.
3. $r \geq \dim F^* + 1$ and RMP are solved exactly, that is the same assumption that Hearn **et al**. (1985) used to can prove that the convergence is finite for RSD. We have introduced the possibility for a restricted simplicial decomposition scheme, that is $r = \infty$, that the RMP's are solved by truncation but at the limit are solved exactly.

García **et al**. (1999b) conduct numerical experiments on non-linear single- and multicommodity network flow problems with various realization of the framework, and compare their performances with those of the NSD and RSD algorithms.

Now, it is described two applications, where the above methodology may be used to solve the sybmodels that result when the design variables are fixed. First a centralized NDP and second a bilevel NDP are presented.

A large number of other applications could be mentioned**.** The CG/SD methodology is special relevant in the context of the traffic assignment models. See Patriksson (1994). The two mentioned applications have been selected because of we have a direct experience and they characterize the NDP with 1 and 2 levels approaches.

4. RAIL FREIGHT PROBLEM

In the medium range planning of the activities in a Railway Freight Transportation System (RFTS), one of the most relevant problems is to define the service level, that is the different train itineraries, called *services*, and their frequencies (number of trips in the time horizon). This is also called *tactical planning RFNDP.* See Salmerón (1998) for more details.

A RFTS is defined by a physical network, where the nodes correspond to stations and the arcs to railway tracks, and by a set of Origin Destination pairs, for each of which and for each type of good to be shipped the demand is known. A service network is defined in term of the basic and classification (car grouping) services. These last ones are associated to a pair of classification yards.

We consider a *commodity*, w, as defined by an Origin/Destination pair and by one type of good; so the number of commodities is bounded by the number of O/D pairs times the number of distinct goods. The demand, b, is given in *carloads*, i.e. in number of cars, per unit of time. The demand is met by the car flow of each commodity w, x^w.

The objective of the tactical planning is to determine, within the planning horizon, the train services to be run and their frequencies, y, (i.e. number of runs within the time horizon), so that the operative and waiting costs associated to the demand flow, f(x), and the design costs (train operative and train make up at the yards, given in function of the cost per unit of frequency on the services, r) are minimized.

Each car flow (x^w) commodity must be non negative and continuous. They must verify the network equilibrium conditions and the sum of de commodity flows for each arc (x) must be inferior or equal to the arc capacity, which is given by the train capacity (q) multiplied by the frequency.

The previous definitions permit us to set a tactical RFNDP that may be identified like an example of a centralized *Network Flow Design Problem* (NFDP), where the routing and the design decisions are taken by a centralized authority. A mixed integer NFDP is defined by:

$$\underset{\substack{x^w \geq 0 \\ y \in Y \subset B}}{\text{Min}} \; f(x) + r^T y,$$

subjecto to $\quad Ax^w = b^w, \; \forall w$

$$x = \sum_w X^w \leq q^T y$$

Using Metaheuristics, Marín and Salmerón (1996) define a decomposition schemes, which submodels are *Capacitated Multicommodity Network Flow Problems*, where the capacity constraints may be relaxed into the objective functon, so the new submodels are *Multicommodity Network Flow Problems* and they may be solved using CG/SD methodology class.

5. MULTICOMMODITY INTERCHANGE DESIGN

In the UMINDP two subproblems must be studied:

- The network equilibrium subproblem: that is to say how the users choose the mode of transport, the interchange and the route on the system of transport. To provide a modeling framework for the consideration of combined mode trips, it is important to decide which choices are modeled by the mode choice model (demand) and which by the route choice on the transport network (supply). In this stage equilibrium between sypply and demand leads to an optimal modal split and traffic assignment.

- The design-subproblem; that is the choice of the location where the interchange will be established, the choice of the dimension of its parking lot and fares, and the type of service of the secondary transit to the interchange.

The decision-maker, usually an authority, evaluates the cost-benefit of the effective state of the transportation system, which depends on the demand at the interchanges. Knowing the network equilibrium, the decision-maker generates a new plan on intervention in the system. This consists of making ot two types of decisions.

The first ones are strategical decisions about the topology of the main transit network and the choice of the type of feeders. The second ones are tactical about the dimensioning and the fare policy of the parking lots.

The demand allocation problem is formulated by means of an equilibrium model with combined modes, where the demand decisions are modeled by nested logit function and the transportation network has been simplified to a transportation model. The model describes the user behavior as a function of the generalized transportation costs.

The UMINDP is formulated by means of a mix integer non-linear oprimization model. Both models have been integrated using bi-level mathematical programming.

$$\text{ULM}(g) : \underset{x \in X}{\text{Min}} \, \Psi /(x, g)$$

$$\text{LLM}(x) : g = \underset{q \in \Omega(x)}{\text{arc.min}} \, T(q, x) \,,$$

where g is the demand share and x is the interchange design variable.

In the lower level model (LLM) is minimized the demand cost objective function, T, which depend on the interchange capacity, fares, and the set of feasible solutions, $\Omega(x)$, also depends on the interchange location design.

In the upper level model (ULM) is taken strategical and tactical design decisions. At strategic level, the location of the interchanges, y, and design of the secondary bus feeder lines z, are studied. At tactical level, the interchange parking capacity, u, and the parking fare, v must be decided. We have denoted as x the set of design variables associated with the decisions of the upper level: x: = (y,z,u,v).

For ULM has been used an objective function, ψ, which is defined as the difference between the cost and the benefit in the transport system. To define the objective function all economic and non-economic factors are changed into monetary worth.

We assume that the decision-maker has a criterion for the evaluation of the level of service. We consider that this consists of two components, first a measure of the effective state of the transport system, which depends on the demand variable g. Second, it also depends on the parking fare, v. This benefit relation will be denoted as B(g,v).

The cost has three components: the location cost, L(y), which is a fixed cost of opening an interchange; the cost of the parking installation and management, P(u), which depends on its capacity; and the third one is the design cost of the secondary transit network, R(y,z), which is a function of the strategic design variables.

$$\psi(x,g) = L(y) + R(y,z) + P(u) - B(g,v).$$

The upper level feasible set, X, is defined considering that each decision has its own constraint set and the strategic variables are binary, y, and integer, z, meanwhile, the tactical decisions, u, v, are continuous and non negative.

García and Marín (1999) solve the above bilevel model using some heuristic based on Simulated Annealing and Greedy techniques, the submodels may be *Combined Multicommodity Network Flow Problems* and they may be solved with the help of CG/SD.

REFERENCES

AHUJA, R.K.; T.L. MAGNANTI and J.B. ORLIN (1993): **Network flows, algorithms, and applications**, Prentice-Hall.

AHUJA, R.K.; T.L. MAGNANTI; J.B. ORLIN and M.R. REDDY (1995): Applications of network optimization, In M.O. Ball, T.L. Magnanti, C.L. Monma and G.L. Nemhauser (eds), **Network models, handbooks of operations research and management science**, Elsevier.

BALAKRISHNAN, A.; T.L. MAGNANTI; A. SHULMAN and R.T. WONG (1991): "Models for planning capacity expansion in local access telecommunication networks", **Annals of Operations Research**, 33, 239-284.

BERTSEKAS, D.P. (1982): "Projected Newton methods optimization problems with simple constraints", **SIAM Journal on Control and Optimization**, 20, 221-246.

CHANG, S. and B. GAVISH (1993): "Telecommunications network topological design and capacity expansion: formulations and algorithms", **Telecommunication Systems**, 1, 99-131.

CRAINIC, T.; J.A. FERLAND and J.M. ROSSEAU (1986): "Multicommodity, multimode freight transportation: a general modeling and algorithmic framework for the service network design problem", **Transportation Research** B, 20B, 225-242.

CRAINIC, T.and J.M. ROSSEAU (1984): "A tactical planning model for rail transportation", **Transportation Science**, 18, 165-184.

FRIESZ, T.; H.J. CHO; N. MEHTA; R. TOBIN and G. ANANDALINGAM (1992): "A Simulated Annealing Approach to the Network Design Problem with Variational Inequality Constraints", **Transportation Science**, 26, 18-26.

GARCIA, R.; A. MARIN and M. PATRIKSSON (1999a): "A class of column generation/simplicial decomposition algorithms in convex differentiable optimization, I: Convergence analysis", **Departamento de Matemática Aplicada y Estadística**, Universidad Politécnica de Madrid.

____ (1999b): "A class of column generation/simplicial decomposition algorithms in convex differentiable optimization, II: Numerical tests", **Departamento de Matemática Aplicada y Estadística**, Universidad Politécnica de Madrid.

GARCIA, R. and A. MARIN (1999): "Urban multimodal interchange design methodology", **Proceedings of the 7th EURO-Working Group Meeting on Transportation**, August 2-6, Helsinki University of Technology, Espoo, Finland.

GENDRON, B. and T.G. CRAINIC (1994): "Relaxation for multicommodity capacitated network design problems", **Publication CRT-965**, Centre de recherche sur les transports, Université de Montréal.

GENDRON, B.; T.G. CRAINIC and A. FRANGIONI (1998): "Multicommodity capacitated network design", **Telecom Network Planning**, editors B. Sansó and P. Soriano, Kluwer Acad. Pub.

HEARN, D.W.; S. LAWPHONGPANICH and J.A. VENTURA (1985): "Finiteness in restricted simplicial decomposition", **Operations Research Letters**, 4, 125-130.

____ (1987): "Restricted simplicial decomposition: computation and extensions", **Mathematical Programming Study**, 3, 99-118.

HERRMANN, J.W.; G. IOANNOU; I. MINIS and J.M. PROTH (1996): "A dual ascent approach to the fixed-charge capacitated network design problem", **European Journal of Operational Research**, 95, 476-490.

HOLLOWAY, C.A. (1974): "An extension of the Frank and Wolfe method of feasible directions", **Mathematical Programming**, 6, 14-27.

HOLMBERG, K. and D. YUAN (1996): "A lagrangian heuristic based branch-and-bound approach for the capacitated network design problem", **Working Paper Department of Mathematics, Linköping Institute of Technology**, Kluwer Acad. Pub., New York.

KEATON, M.H. (1992): "Designing railroad operating plans: a dual adjustment method for implementing Lagrangean relaxation", **Transportation Science** 26(4), 263-279.

LAGUNA, M. (1998): "Applying robust optimization to capacity expansion of one location in telecommunications with demand uncertainty", **Management Science**, 44, 11, part 2 of 2, S101-S110.

LARSSON, M.; M. PATRIKSSON and C. RYDERGREN (1996): "Simplicial decomposition with nonlinear column generation", report, Department of Mathematics, Linköping Institute of Technology, Linköping, Sweden.

____ (1997): "Applications of simplicial decomposition with nonlinear column generation to nonlinear network flows", in Network Optimization, Pardalos, P.M., Hager, W.W. and Hearn, D.W., eds., vol. 450 of Lectures Notes in Economics and Mathematical Systems, Springer-Verlag, Berlin, 346-373.

MAGNANTI, T.L. and R.T. WONG (1984): "Network design and transportation planning: models and algorithms", **Transportation Science** 18(1), 1-55.

MARIN, A. and J. SALMERON (1996): "Tactical design of rail freight networks, Part I: Exact and heuristic methods", **Eur. Journal of Op. Res.**, 90, 26-44.

MINOUX, M. (1989): "Network synthesis and optimum network design problems; models, solution methods and applications", **Networks**, 19, 313-360.

OPPENHEIM, N. (1994): **Urban Travel Demand Modelling**, Wiley-Interscience, New York.

PARRISH, S.H.; T. COX; W. KUEHNER and Y. QIU (1992): "Planning for optimal expansion of leased line communication networks", **Annals of Op. Res**., 36, 347-364.

PATRIKSSON, M. (1994): **The Traffic Assignment Problem, Models and Methods**, VSP, Utrecht, The Netherlands.

_______________ (1999): **Nonlinear Programming and Variational Inequality Problems**, Kluwer Academic Publishers, New York.

RAJAGOPALAN, S.; M.R. SINGH and T.E. MORTON (1998): "Capacity expansion and replacement with uncertain technological breakthroughs", **Manag. Science**, 44(1), 12-30.

SALMERON, J. (1998): "Optimización en diseño de grandes sistemas", Doctoral Thesis, Departamento de Matemática Aplicada y Estadística, Universidad Politécnica de Madrid.

SHULMAN, A. and R. VACHANI (1993): "Decomposition algorithm for capacity expansion of local access networks", **IEEE Transactions on Communications**, 41(7), 1063-1073.

VON HOHENBALKEN, B. (1977): "Simplicial decomposition in nonlinear programming algorithms", **Mathematical Programming**, 13, 46-68.

ZANGWILL, W.I. (1969): **Nonlinear Programming: A Unified Approach**, Prentice-Hall, Englewood Cliffs, NJ.