ON CHARACTERIZING MAXIMAL COVERS

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ABSTRACT
In this paper we introduce the concept of maximal covers and provide some characterizations that make the identification of the maximal covers from the set of covers implied by a 0-1 knapsack constraint easier. By construction, these maximal covers induce non-dominated valid inequalities for the set of feasible solutions for the Knapsack constraint. So, their identification can help to tightening 0-1 models. We also show some situations where a procedure taken from the literature for identifying non-dominated inequalities from certain types of covers only obtains a small subset of maximal covers.

Key words: maximal covers, tighter formulations, knapsack constraints, dominated inequalities.

MSC: 90C10

RESUMEN
En este trabajo se introduce el concepto de cubrimiento maximal y se proporcionan algunas caracterizaciones que facilitan la identificación de los cubrimientos maximales respecto del conjunto de cubrimientos implicados por una restricción de tipo mochila con variables 0-1. Por construcción, estos cubrimientos maximales inducen desigualdades no dominadas que son válidas para el conjunto de soluciones factibles para la restricción de tipo mochila. Así pues, su identificación puede contribuir al reforzamiento de modelos 0-1. Asimismo, se muestran algunas situaciones en las que un procedimiento descrito en la literatura que identifica desigualdades no dominadas a partir de ciertos tipos de cubrimientos únicamente obtiene un pequeño subconjunto de cubrimientos maximales.

Palabras clave: tapas máximas, formulaciones más firmes, constreñimiento de la mochila, desigualdades dominadas,

1. INTRODUCTION

Consider the 0-1 linear programming problem

$$\max\left\{\sum_{j\in J} c_j x_j \;\middle|\; \sum_{j\in J} a_{ij} x_j \le b_i \quad \forall i \in I,\ \ x_j \in \{0,1\} \quad \forall j \in J\right\}, \tag{1}$$

where J = {1,...,n} and I = {1,...,m}.

The LP relaxation of (1) is the same problem (1) where each variable x_j is allowed to take any value in the interval [0,1].

We say that two constraint systems $\mathbf{Ax} \le \mathbf{b}$ and $\mathbf{A'x} \le \mathbf{b'}$ are equivalent if they admit exactly the same set of 0-1 solutions. The system $\mathbf{A'x} \le \mathbf{b'}$ is said to be as tight as the system $\mathbf{Ax} \le \mathbf{b}$ if it is equivalent to $\mathbf{Ax} \le \mathbf{b}$ and $\{\mathbf{x} \in [0,1]^n \mid \mathbf{A'x} \le \mathbf{b'}\} \subseteq \{\mathbf{x} \in [0,1]^n \mid \mathbf{Ax} \le \mathbf{b}\}$.The system $\mathbf{A'x} \le \mathbf{b'}$ is said to be tighter than the system $\mathbf{Ax} \le \mathbf{b}$ if it is equivalent to $\mathbf{Ax} \le \mathbf{b}$ and $\{\mathbf{x} \in [0,1]^n \mid \mathbf{A'x} \le \mathbf{b'}\} \subset \{\mathbf{x} \in [0,1]^n \mid \mathrm{A}\mathbf{x} \le \mathbf{b}\}$.

We say that the inequality $\sum_{j=1}^{n} a_j x_j \le b$ is valid for a set $R \subseteq \mathrm{IR}^n$ if it is satisfied by any vector $(x_1,...,x_n) \in R$.

The tighter a 0-1 model, the smaller could the gap be between the optimal values of the related 0-1 problem and its LP relaxation, and, less computational effort could be required to solve the problem. Thus, we

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are interested in finding tighter formulations for problem (1), see {Dietrich et.al. (1993), Hoffmann-Padberg (1991), Padberg (1975), Savelsberg (1994)] among others. Obviously, a scheme for obtaining as-tight-as formulation for (1) consist of appending to the constraint system of (1) valid inequalities for its feasible region. It is well known that the scheme can be more effective by appending valid inequalities that have been violated by feasible solutions for the LP relaxation of (1). See Johnson **et al**.(2000) for a good survey on the subject.

The main contribution of this paper is the introduction of the concept of maximal covers and its related characterization. This concept is defined as a generalization of the concept of maximal cliques, see Muñoz (1999). A maximal cover from a set of covers is a cover whose induced inequality is not dominated by the inequality induced by any other cover from the set. The inequalities induced by the maximal covers from the set of covers implied by a knapsack constraint of problem (1) are valid for its feasible region, and they can be used in constraint reformulation. We show that these maximal covers can be characterized as the extensions of certain minimal covers.

This paper is organized as follows: Sections 2 and 3 review classical types of covers and introduce the concept of maximal covers. A theoretical background is also given to state in Section 4 a characterization of those valid inequalities. Section 5 draws some conclusions from this work.

2. COVERS IMPLIED BY A CONSTRAINT

Given a set of variables $\{x_1,...,x_n\}$ and a set $F \subseteq \{1,...,n\}$, let us denote $X(F) \equiv \sum_{j \in F} x_j$.

See e.g. [12] for more details about some of the concepts defined throughout the paper. See also the references compiled in Aardal-Weismantel (1997).

Definition 1. A **cover** C is a set of indices of variables that can be expressed as the union of two disjoint sets, say C^+ and C^-, such that

$$X(C^+) - X(C^-) \leq k_c - |C^-|, \tag{2}$$

where k_c is an integer with $1 \leq k_c \leq |C|$. Inequality (2) is said to be **induced** by C.

Definition 2. A **trivial cover** is a cover C such that $k_c = |C|$.

Definition 3. A cover C with induced inequality (2) is **implied** by the inequality $\sum_{j=1}^{n} a_j x_j \leq b$ if (2) is a valid inequality for $\{(x_1,...,x_n) \in \{0,1\}^n \mid \sum_{j=1}^{n} a_j x_j \leq b\}$.

We consider knapsack constraints from problem (1) of the form

$$\sum_{j \in J_0} a_j x_j \leq b, \tag{3}$$

where $0 < a_j \leq b \ \ \forall j \in J_0$ and $a_j \leq a_{j'} \ \ \forall j, j' \in J_0$ with $j < j'$. (Note that any constraint in 0-1 variables can be put in this form). Without loss of generality, we assume that $\sum_{j \in J_0} a_j > b$. Let F_0 be the set of 0-1 solutions that satisfy constraint (3), i.e., $F_0 = \{(x_j)_{j \in J} \in \{0,1\}^n \mid \sum_{j \in J_0} a_j x_j \leq b\}$.

Lemmas 1 and 2 state some necessary and sufficient conditions for a subset of J to be a non-trivial cover implied by constraint (3). These conditions lead to the characterization given in Proposition 1, which will be used to state in Theorem 1 a necessary and sufficient condition for certain covers to be implied by constraint (3).

Lemma 1. Let C^+, $C^- \subseteq J$ be two disjoint sets with non-empty union and let k_c be an integer such that $k_c \leq |C^+| + |C^-| - 1$. If $X(C^+) - X(C^-) \leq k_c - |C^-|$ is a valid inequality for F_0, then

(1) $|C^+ \cap J_0| \geq 2$.

(2) $k_c \geq |C^+ \setminus J_0| + |C^-| + 1$.

(3) $\sum_{j \in F} a_j > b \quad \forall F \subseteq C^+$ with $|F| = k_c - |C^-| + 1$.

Proof. (1) Suppose that $|C^+ \cap J_0| \leq 1$ and let $x_j = 1\ \forall j \in C^+$ and $x_j = 0\ \forall j \in J \setminus C^+$. Since $b > 0$, $a_j \leq b\ \forall j \in J_0$ and $|C^+| \geq k_c - |C^-| + 1$, we have that $\sum_{j \in J_0} a_j x_j = \sum_{j \in C^+ \cap J_0} a_j \leq b$ and $\sum_{j \in C^+} x_j - \sum_{j \in C^-} x_j = |C^+| > k_c - |C^-|$, which contradicts the fact that $X(C^+) - X(C^-) \leq k_c - |C^-|$ is a valid inequality for F_0.

(2) Suppose that $k_c < |C^+ \setminus J_0| + |C^-| + 1$ and let $k \in C^+ \cap J_0$, $x_j = 1\ \forall j \in (C^+ \setminus J_0) \cup \{k\}$ and $x_j = 0\ \forall j \in J \setminus ((C^+ \setminus J_0) \cup \{k\})$. Then $\sum_{j \in J_0} a_j x_j = a_k \leq b$ and $\sum_{j \in C^+} x_j - \sum_{j \in C^-} x_j = |C^+ \setminus J_0| + 1 > k_c - |C^-|$, contradicting the initial hypothesis.

(3) Let $F \subseteq C^+$ be such that $|F| = k_c - |C^-| + 1$ and suppose that $\sum_{j \in F} a_j \leq b$. Choosing $x_j = 1\ \forall j \in F$ and $x_j = 0\ \forall j \in J \setminus F$ it results $\sum_{j \in J_0} a_j x_j = \sum_{j \in F} a_j \leq b$ and $\sum_{j \in C^+} x_j - \sum_{j \in C^-} x_j = |F| > k_c - |C^-|$, which is a contradiction.

Lemma 2. Let $C^+, C^- \subseteq J$ be two disjoint sets and let k_c be an integer such that $k_c \leq |C^+| + |C^-| - 1$. If $k_c \geq |C^-| - 1$ and $\sum_{j \in F} a_j > b\ \forall F \subseteq C^+$ with $|F| = k_c - |C^-| + 1$, then $X(C^+) - X(C^-) \leq k_c - |C^-|$ is a valid inequality for F_0.

Proof. Let $(x_j)_{j \in J} \in \{0,1\}^n$ and $C_1^+ = \{j \in C^+ | x_j = 1\}$. If $\sum_{j \in C^+} x_j - \sum_{j \in C^-} x_j > k_c - |C^-|$, then $\exists\ F \subseteq C_1^+$ such that $|F| = k_c - |C^-| + 1$, since $|C_1^+| = \sum_{j \in C^+} x_j \geq \sum_{j \in C^+} x_j - \sum_{j \in C^-} x_j$ and $k_c - |C^-| + 1 \geq 0$. Therefore $\sum_{j \in J_0} a_j x_j \geq \sum_{j \in F} a_j x_j = \sum_{j \in F} a_j > b$.

Proposition 1. Let $C \subseteq J$ be a non-trivial cover with induced inequality (2).

Then C is implied by constraint (3) if and only if $k_c \geq |C^+ \setminus J_0| + |C^-| + 1$ and $\sum_{j \in F} a_j > b \quad \forall F \subseteq C^+$ with $|F| = k_c - |C^-| + 1$.

Proof. It follows from claims (2) and (3) of Lemma 1 and from Lemma 2.

Lemma 3. Let $C \subseteq J$ and I be an integer such that $|C \setminus J_0| + 2 \leq I \leq |C|$. Then the following statements are equivalent:

(1) $\sum_{j \in F} a_j > b \ \ \forall F \subseteq C$ with $|F| = I$.

(2) $\sum_{j \in F} a_j > b \quad \forall F \subseteq C \cap J_0$ with $|F| = I - |C \setminus J_0|$.

Proof. (1) $\Rightarrow$ (2) Let $F \subseteq C \cap J_0$ be such that $|F| = l - |C \setminus J_0|$. Since $\sum_{j \in F} a_j = \sum_{j \in F \cup (C \setminus J_0)} a_j$, $F \cup (C \setminus J_0) \subseteq C$ and $|F \cup (C \setminus J_0)| = l$, we have that $\sum_{j \in F} a_j > b$.

(2) $\Rightarrow$ (1) Let $F \subseteq C$ be such that $|F| = l$. Them $\exists$ $F' \subseteq F \cap J_0$ with $|F'| = l - |C \setminus J_0|$, since $|F \cap J_0| = |F| - |F \setminus J_0| = l - |F \setminus J_0| \geq l - |C \setminus J_0| > 0$. So, $\sum_{j \in F} a_j \geq \sum_{j \in F'} a_j > b$.

Given a non-empty set $C \subseteq J_0$, let $m_l(C)$ be the set of the l smallest indices of C, where l is an integer such that $1 \leq l \leq |C|$, and let $\underline{\gamma}(C) = \min\{j \mid j \in C\}$ and $\overline{\gamma}(C) = \max\{j \mid j \in C\}$.

Proposition 2. Let $C \subseteq J$ be a non-trivial cover with induced inequality (2).

Then C is implied by constraint (3) if and only if $k_c \geq |C^+ \setminus J_0| + |C^-| + 1$ and

$$\sum_{j \in m_{k_c - |C^+ \setminus J_0| - |C^-| + 1}(C^+ \cap J_0)} a_j > b.$$

Proof. It follows from Proposition 1 and Lemma 3, since $a_j \leq a_{j'}$ $\forall j, j' \in J_0$ with $j < j'$.

Theorem 1. Let $C \subseteq J$ be a cover with induced inequality (2) such that $C^+ \cap J_0 \neq \phi$.

Then C is implied by constraint (3) if and only if $k_c \geq |C^+ \setminus J_0| + |C^-| + l_c$, where $l_c = \max\{l \mid \sum_{j \in m_l(C^+ \cap J_0)} a_j \leq b\}$.

Proof. ($\Rightarrow$) If C is a trivial cover, it results $k_c = |C^+ \cap J_0| + |C^+ \setminus J_0| + |C^-| \geq |C^+ \setminus J_0| + |C^-| + l_c$.

If C is a non-trivial cover, by Proposition 2 we have that $\sum_{j \in m_{k_c - |C^+ \setminus J_0| - |C^-| + 1}(C^+ \cap J_0)} a_j > b$, hence $l_c \leq k_c - |C^+ \setminus J_0| - |C^-|$.

($\Leftarrow$) If C is a trivial cover, it is clear that C is implied by constraint (3). If C is a non-trivial cover, it follows that $k_c \geq |C^+ \setminus J_0| + |C^-| + 1$ and $\sum_{j \in m_{k_C - |C^+ \setminus J_0| - |C^-| + 1}(C^+ \cap J_0)} a_j > b$, since $1 \leq l_c \leq k_c - |C^+ \setminus J_0| - |C^-|$; thus, by Proposition 2, C is implied by constraint (3).

3. MAXIMAL COVERS FROM A SET OF COVERS

Definition 4. The inequality $\sum_{j=1}^{n} a_j x_j \leq b$ is **dominated** by the inequality $\sum_{j=1}^{n} a'_j x_j \leq b'$ if $\{(x_1,\ldots,x_n) \in [0,1]^n \mid \sum_{j=1}^{n} a'_j x_j \leq b'\} \subseteq \{(x_1,\ldots,x_n) \in [0,1]^n \mid \sum_{j=1}^{n} a_j x_j \leq b\}$.

Definition 5. Given a set of covers C, C $\in$ C is a **maximal cover** from C if its induced inequality is not dominated by the inequality induced by C' $\forall$C' $\in$ C such that $C'^+ \neq C^+$ or $C'^- \neq C^-$ or $k_{c'} \neq k_c$.

There are several ways for tightening the formulation of problem (1) by using maximal covers whose induced inequalities are valid for its feasible region. It can be done by appending those induced inequalities to the constraint system of (1) (e.g., provided that they are violated by the optimal solution of its LP relaxation) and by increasing or reducing the coefficients of some constraints in (1), [see Dietrich **et al**. (1993), Escudero **et al**. (1998), Escudero-Muñoz (1998), Muñoz (1999)] among others.

Example 1 illustrates he case where the coefficient reduction method proposed in (4) [see also Escudero-Muñoz (1998), Muñoz (1999)] can be applied to a knapsack constraint by using a maximal cover from the set of covers implied by that constraint, but it cannot be applied by using a non-maximal cover.

Example 1. Consider the 0-1 knapsack constraint

$$x_1 + x_2 + 4x_3 + 7x_4 + 8x_5 + 9x_6 \leq 10 \tag{4}$$

Consider the covers C = {3, 4, 5, 6} and C' = {3, 4, 5}, and let (5) and (6) be their induced inequalities, respectively.

$$x_3 + x_4 + x_5 + x_6 \leq 1 \tag{5}$$

$$x_3 + x_4 + x_5 \leq 1 \tag{6}$$

It can be shown that C is a maximal cover from the set of covers implied by constraint (4), see Theorem 3. On the other hand, C' is implied by (4), but it is not a maximal cover from the set of covers implied by (4), since inequality (6) is dominated by inequality (5).

By applying a coefficient reduction approach (see Dietrich-Escudero-Chance (1993) to constraint (4) using the cover C, we obtain the constraint

$$x_1 + x_2 - 4x_3 - x_4 + x_6 \leq 2 \tag{7}$$

(See that the constraint system given by (7) and (5) is tighter than constraint (4)).

On the contrary, the coefficients of constraint (4) cannot be reduced by using the cover C'.

Based on Definition 5, identifying maximal covers from the set of covers implied by constraint (3) can be a hard task. It is useful to introduce some characterizations that make easier the identification of such covers.

Proposition 3. Let $X(C^+) - X(C^-) \leq k_C - |C^-|$ and $X(C'^+) - X(C'^-) \leq k_{C'} - |C'^-|$ be the inequalities induced by the covers C and C' respectively. If $k_C - k_{C'} \geq |C^+ \setminus C'^+| + |C^- \setminus C'^-|$, then the inequality induced by C is dominated by the inequality induced by C'.

Proof. Let $(x_j)_{j\in C\cup C'} \in [0,1]^{|C\cup C'|}$ be such that $\sum_{j\in C'^+} x_j - \sum_{j\in C'^-} x_j \leq k_{C'} - |C'^-|$.

Considering that $\sum_{j\in C'^+\cap C^+} x_j \leq \sum_{j\in C'^+} x_j$ and $C'^- = (C'^- \cap C^-) \cup (C'^- \setminus C^-)$, it results $\sum_{j\in C'^+\cap C^+} x_j - \sum_{j\in C'^-\cap C^-} x_j \leq k_{C'} - (|C'^-| - |C'^- \setminus C^-|) = k_{C'} - (|C^-| - |C^- \setminus C'^-|)$.

Therefore we have that $\sum_{j\in C^+} x_j - \sum_{j\in C^-} x_j \leq k_{C'} - |C^-| + |C^- \setminus C'^-| + |C^+ \setminus C'^+| \leq k_C - |C^-|$, since $\sum_{j\in C^+} x_j - \sum_{j\in C^-} x_j = \sum_{j\in C^+\cap C'^+} x_j + \sum_{j\in C^+\setminus C'^+} x_j - \sum_{j\in C^-\cap C'^-} x_j - \sum_{j\in C^-\setminus C'^-} x_j$.

Proposition 4. Let $X(C^+) - X(C^-) \leq k_C - |C^-|$ and $X(C'^+) - X(C'^-) \leq k_{C'} - |C'^-|$ be the inequalities induced by the covers C and C' respectively. If C is a non-trivial cover and its induced inequality is dominated by the inequality induced by C', then C' is a non-trivial cover and $k_C - k_{C'} \geq |C^+ \setminus C'^+| + |C^- \setminus C'^-|$.

Proof. The result will follow if it can be shown that $\exists (x_j)_{j\in C\cup C'} \in [0,1]^{|C\cup C'|}$ such that $\sum_{j\in C'^+} x_j - \sum_{j\in C'^-} x_j \le k_{C'} - |C'^-|$ and $\sum_{j\in C^+} x_j - \sum_{j\in C^-} x_j \ge k_{C'} + |C^+ \setminus C'^+| - |C^- \cap C'^-|$. In this case we have that $k_C - |C^-| \ge k_{C'} + |C^+ \setminus C'^+| - |C^- \cap C'^-|$, from which $k_C - k_{C'} \ge |C^+ \setminus C'^+| + |C^- \setminus C'^-|$; so, C' is a non-trivial cover, since $k_{C'} \le |C^+| + |C^-| - 1 - |C^+ \setminus C'^+| - |C^- \setminus C'^-| = |C^+ \cap C'^+| + |C^- \cap C'^-| - 1 \le |C'| - 1$.

- If $k_{C'} \ge |C^- \cap C'^-|$, suppose that $|C^+ \cap C'^+| \le k_{C'} - |C^- \cap C'^-|$ and let $x_j = 1 \ \forall j \in C^+ \cup (C'^- \setminus C)$ and $x_j = 0 \ \forall j \in C^- \cup (C'^+ \setminus C)$. Considering that $C'^+ = (C'^+ \cap C) \cup (C'^+ \setminus C)$, $C'^- = (C'^- \cap C^-) \cup (C'^- \setminus C^-)$ and C is a non-trivial cover, it results $\sum_{j\in C'^+} x_j - \sum_{j\in C'^-} x_j = |C'^+ \cap C^+| - |C'^- \setminus C^-| \le k_{C'} - |C'^- \cap C^-| - |C'^- \setminus C^-| = k_{C'} - |C'^-|$ and $\sum_{j\in C^+} x_j - \sum_{j\in C^-} x_j = |C^+| > k_C - |C^-|$, contradicting the initial hypothesis. Consequently, it must be $|C^+ \cap C'^+| > k_{C'} - |C^- \cap C'^-|$ and, thus, $\exists\ F \subseteq C^+ \cap C'^+$ with $|F| = k_{C'} - |C^- \cap C'^-|$. Now, let $x_j = 1 \ \forall j \in F \cup (C^+ \setminus C'^+) \cup (C'^- \setminus C^-)$ and $x_j = 0 \ \forall j \in (C'^+ \setminus F) \cup C^-$. Since $C'^+ = F \cup (C'^+ \setminus F)$, $C'^- = (C'^- \cap C^-) \cup (C'^- \setminus C^-)$ and $C^+ = (C^+ \cap C'^+) \cup (C^+ \setminus C'^+)$, it follows that $\sum_{j\in C'^+} x_j - \sum_{j\in C'^-} x_j = |F| - |C'^- \setminus C^-| = k_{C'} - |C'^-|$ and $\sum_{j\in C^+} x_j - \sum_{j\in C^-} x_j = |F| + |C^+ \setminus C'^+| = k_{C'} + |C^+ \setminus C'^+| - |C^- \cap C'^-|$.

- If $k_{C'} < |C^- \cap C'^-|$, we have that $0 < |C^- \cap C'^-| - k_{C'} < |C^- \cap C'^-|$. Accordingly, $\exists\ F \subseteq C^- \cap C'^-$ with $|F| = |C^- \cap C'^-| - k_{C'}$ and, choosing $x_j = 1 \ \forall j \in (C^+ \setminus C'^+) \cup F \cup (C'^- \setminus C)$ and $x_j = 0 \ \forall j \in C'^+ \cup (C^- \setminus F)$, it results $\sum_{j\in C'^+} x_j - \sum_{j\in C'^-} x_j = -\left(|C'^- \setminus C^-| + |F|\right) = k_{C'} - |C'^-|$ and $\sum_{j\in C^+} x_j - \sum_{j\in C^-} x_j = |C^+ \setminus C'^+| - |F| = k_{C'} + |C^+ \setminus C'^+| - |C^- \cap C'^-|$.

Corollary 1. Let $X(C^+) - X(C^-) \le k_C - |C^-|$ and $X(C'^+) - X(C'^-) \le k_{C'} - |C'^-|$ be the inequalities induced by the covers C and C' respectively. If C is a non-trivial cover and any of the following conditions is satisfied, then the inequality induced by C is not dominated by the inequality induced by C'.

(1) $k_{C'} = |C'|$.

(2) $k_C < k_{C'}$.

(3) $k_C = k_{C'}$ and $C^+ \not\subseteq C'^+$.

(4) $k_C = k_{C'}$ and $C^- \not\subseteq C'^-$.

Proposition 5 provides a procedure for obtaining some covers implied by constraint (3) whose induced inequalities dominate the inequality induced by a given cover implied by (3).

Proposition 5. Let C be a cover implied by constraint (3) with induced inequality (2).

If $C^+ \cap J_0 \neq \phi$, then for any pair of sets C'^+, C'^- and any integer $k_{C'}$ such that $C^+ \cap J_0 \subseteq C'^+ \subseteq C^+$, $C'^- \subseteq C^-$ and $|C'^+ \setminus J_0| + |C'^-| + l_C \le k_{C'} \le k_C - |C^+ \setminus C'^+| - |C^- \setminus C'^-|$, where $l_C = \max\{l | \sum_{j\in m_l(C^+\cap J_0)} a_j \le b\}$, the following properties hold:

(1) $C'^+ \cup C'^-$ is a cover implied by constraint (3), and inequality (8) is induced by $C'^+ \cup C'^-$.

$$X(C'^+) - X(C'^-) \le k_{C'} - |C'^-| \tag{8}$$

(2) If C is a non-trivial cover, then $k_{C'} \le |C'^+| + |C'^-| - 1$.

(3) Inequality (2) is dominated by inequality (8).

Proof. (1) Since $C'^+ \cap J_0 = C^+ \cap J_0$, we have that $C'^+ \cap J_0 \neq \phi$ and $k_{C'} \geq |C'^+ \setminus J_0| + |C'^-| + \max\{l | \sum_{j \in m_l(C'^+ \cap J_0)} a_j \leq b\}$. On the other hand, considering that $k_C \leq |C^+| + |C^-|$ it results $k_{C'} \leq |C^+| - |C^+ \setminus C'^+| + |C^-| - |C^- \setminus C'^-| = |C'^+| + |C'^-|$. Hence, by Theorem 1, $C'^+ \cup C'^-$ is a cover implied by constraint (3) and inequality (8) is induced by it.

(2) It follows from the proof of claim (1) above.

(3) It follows from Proposition 3.

Note. Given a cover implied by constraint (3) with induced inequality (2), it is obvious that there exist two sets, say C'^+ and C'^-, such that $C^+ \cap J_0 \subseteq C'^+ \subseteq C^+$ and $C'^- \subseteq C^-$. Furthermore, if $C^+ \cap J_0 \neq \phi$, by Theorem 1 we have that $k_C \geq |C^+ \setminus J_0| + |C^-| + l_C$ and, since $|C^+ \setminus J_0| = |C^+ \setminus C'^+| + |C'^+ \setminus J_0|$ and $|C^-| = |C^- \setminus C'^-| + |C'^-|$, it results $|C'^+ \setminus J_0| + |C'^-| + l_C \leq k_C - |C^+ \setminus C'^+| - |C^- \setminus C'^-|$.

Therefore the hypotheses of Proposition 5 are correct.

Proposition 6 states some conditions to be satisfied by a maximal cover from the set of covers implied by constraint (3). These conditions lead to the characterization given in Theorem 2, which will be improved in Theorems 3 and 4.

Proposition 6. Let C be a maximal cover from the set of covers implied by constraint (3). Then C is a non-trivial cover, $C \subseteq J_0$ and its induced inequality is $X(C) \leq \max\{l | \sum_{j \in m_l(C)} a_j \leq b\}$.

Proof. Let (2) be the inequality induced by C. It can easily be verified that C is a non-trivial cover; so, by claim (1) of Lemma 1 it follows that $C^+ \cap J_0 \neq \phi$.Thus, choosing $C'^+ = C^+ \cap J_0$, $C'^- = \phi$ and $k_{C'} = l_C$ in Proposition 5, it results that $C^+ \cap J_0$ is a cover implied by constraint (3), and the inequality $X(C^+) - X(C^-) \leq k_C - |C^-|$ is dominated by the inequality $X(C^+ \cap J_0) \leq l_C$. Consequently, it must be $C^+ = C^+ \cap J_0$, $C^- = \phi$ and $k_C = l_C$, whence $C \subseteq J_0$ and its induced inequality is $X(C) \leq \max\{l | \sum_{j \in m_l(C)} a_j \leq b\}$.

Theorem 2. C is a maximal cover from the set of covers implied by constraint (3) if and only if C is a maximal cover from the set of non-trivial covers $C' \subseteq J_0$ with induced inequality $X(C') \leq \max\{l | \sum_{j \in m_l(C')} a_j \leq b\}$.

Proof. (⇒) It follows from Proposition 6 and Theorem 1.

(⇐) Let C' be a cover implied by constraint (3) with induced inequality $X(C'^+) - X(C'^-) \leq k_{C'} - |C'^-|$ such that $C'^+ \neq C$ or $C'^- \neq \phi$ or $k_{C'} \neq \max\{l | \sum_{j \in m_l(C)} a_j \leq b\}$. By Theorem 1, it suffices to prove that the inequality induced by C is not dominated by the inequality induced by C'.

If C' is a trivial cover, by Corollary 1 the inequality induced by C is not dominated by the inequality induced by C'.

If C' is a non-trivial cover, by claim (1) of Lemma 1 we have that $C'^+ \cap J_0 \neq \phi$.

Let $X(C'^+ \cap J_0) \leq \max\{l | \sum_{j \in m_l(C'^+ \cap J_0)} a_j \leq b\}$ be the inequality induced by $C'^+ \cap J_0$. Then, by Proposition 5, $C'^+ \cap J_0$ is a non-trivial cover implied by constraint (3) and the inequality induced by C' is dominated by the inequality induced by $C'^+ \cap J_0$.

- If $C'^{+} \cap J_0 = C$, the inequality induced by C' is dominated by the inequality induced by C. Therefore, by Proposition 4 it follows that the inequality induced by C is not dominated by the inequality induced by C'.

- If $C'^{+} \cap J_0 \neq C$, the inequality induced by C is not dominated by the inequality induced by $C'^{+} \cap J_0$, since $C'^{+} \cap J_0$ is a non-trivial cover with induced inequality $X(C'^{+} \cap J_0) \leq \max \{l \mid \sum_{j \in m_l(C'^{+} \cap J_0)} a_j \leq b\}$.

Given a maximal cover from the set of covers implied by constraint (3), Proposition 7 shows that either adding another variable to the left-hand-side of its induced inequality, or deleting a variable from the left-hand-side and simultaneously reducing in one unit the right-hand-side, the resulting inequalities are not valid for the set F_0. Proposition 8 shows the converse.

Proposition 7. Let C be a maximal cover from the set of covers implied by constraint (3) with induced inequality $X(C) \leq k_C$. Then the inequalities $X(C \cup \{k\}) \leq k_C$ and $X(C \setminus \{k'\}) \leq k_C - 1$, where $k \in J \setminus C$ and $k' \in C$, are not valid for F_0.

Proof. Trivial.

Proposition 8. Let $C \subseteq J_0$ be a non-trivial cover with induced inequality $X(C) \leq k_C$, where $k_C = \max \{l \mid \sum_{j \in m_l(C)} a_j \leq b\}$. If for all $k \in J_0 \setminus C$ and $k' \in C$ the inequalities $X(C \cup \{k\}) \leq k_C$ and $X(C \setminus \{k'\}) \leq k_C - 1$ are not valid for F_0, then C is a maximal cover from the set of covers implied by constraint (3).

Proof. Let $C' \neq C$ be a non-trivial cover in J_0 with induced inequality $X(C') \leq k_{C'}$, where $k_{C'} = \max \{l \mid \sum_{j \in m_l(C')} a_j \leq b\}$. If it can be shown that $k_C - k_{C'} < |C \setminus C'|$, by Proposition 4 and Theorem 2 it will follow that C is a maximal cover from the set of covers implied by constraint (3).

Indeed, by Theorem 1 it results that $X(C') \leq k_{C'}$ is a valid inequality for F_0.

Suppose that $k_C - k_{C'} \geq |C \setminus C'|$ and let $(x_j)_{j \in J} \in F_0$.

If $C' \not\subseteq C$, let $k \in C' \setminus C$. Since $C \cup \{k\} \subseteq C' \cup (C \setminus C')$, we have that $\sum_{j \in C \cup \{k\}} x_j \leq \sum_{j \in C'} x_j + \sum_{j \in C \setminus C'} x_j \leq k_{C'} + |C \setminus C'| \leq k_C$, which is a contradiction.

If $C' \subset C$, let $k' \in C \setminus C'$. Since $C \setminus \{k'\} = C' \cup ((C \setminus C') \setminus \{k'\})$, it follows that $\sum_{j \in C \setminus \{k'\}} x_j = \sum_{j \in C'} x_j + \sum_{j \in (C \setminus C') \setminus \{k'\}} x_j \leq k_{C'} + |C \setminus C'| - 1 \leq k_C - 1$, which is also a contradiction.

Theorem 3 states a necessary and sufficient condition for a proper subset of J_0 to be a maximal cover from the set of covers implied by constraint (3).

Theorem 3. Let $C \subset J_0$ be a non-trivial cover with induced inequality $X(C) \leq k_C$, where $k_C = \max \{l \mid \sum_{j \in m_l(C)} a_j \leq b\}$. Then C is a maximal cover from the set of covers implied by constraint (3) if and only if

$$\sum_{j \in m_{k_C}(C)} a_j + a_{\bar{\gamma}(J_0 \setminus C)} \leq b \text{ and } \sum_{j \in m_{k_C}(C \setminus \{\underline{\gamma}(C)\})} a_j \leq b.$$

Proof. ($\Rightarrow$) Let $C \subset J_0$ be a maximal cover from the set of covers implied by constraint (3). Then the inequalities $X(C \cup \{\bar{\gamma}(J_0 \setminus C)\}) \leq k_C$ and $X(C \setminus \{\underline{\gamma}(C)\}) \leq k_C - 1$ are not valid for F_0.

Let us consider the inequality $X(C \cup \{\bar{\gamma}(J_0\backslash C)\}) \le k_C$. By Proposition 2, it results $\sum_{j\in m_{k_C+1}(C\cup\{\bar{\gamma}(J_0\backslash C)\})} a_j \le b$; therefore $m_{k_C+1}(C \cup \{\bar{\gamma}(J_0\backslash C)\}) \ne m_{k_C+1}(C)$, since $\sum_{j\in m_{k_C+1}(C)} a_j > b$. Moreover $m_{k_C+1}(C \cup \{\bar{\gamma}(J_0\backslash C)\}) = m_{k_C}(C) \cup \{\min\{\bar{\gamma}(J_0\backslash C), \underline{\gamma}(C\backslash m_{k_C}(C))\}\}$ and $m_{k_C+1}(C) = m_{k_C}(C) \cup \{\underline{\gamma}(C\backslash m_{k_C}(C))\}$; thus, it must be $m_{k_C+1}(C \cup \{\bar{\gamma}(J_0\backslash C)\}) = m_{k_C}(C) \cup \{\bar{\gamma}(J_0 \backslash C)\}$, whence $\sum_{j\in m_{k_C}(C)} a_j + a_{\bar{\gamma}(J_0\backslash C)} \le b$.

Now, let us consider the inequality $X(C \backslash \{\underline{\gamma}(C)\}) \le k_C - 1$. If $k_C = 1$, we have that $\sum_{j\in m_{k_C}(C\backslash\{\underline{\gamma}(C)\})} a_j \le b$

If $k_C > 1$, by Proposition 2 it follows that $\sum_{j\in m_{k_C}(C\backslash\{\underline{\gamma}(C)\})} a_j \le b$.

($\Leftarrow$) Let $C \subset J_0$ be a non-trivial cover with induced inequality $X(C) \le k_C$ such that $\sum_{j\in m_{k_C}(C)} a_j + a_{\bar{\gamma}(J_0\backslash C)} \le b$ and $\sum_{j\in m_{k_C}(C\backslash\{\underline{\gamma}(C)\})} a_j \le b$, and let $k \in J_0 \backslash C$ and $k' \in C$.

By Theorem 1 and Proposition 8, it suffices to prove that the inequalities $X(C \cup \{k\}) \le k_C$ and $X(C \backslash \{k'\}) \le k_C - 1$ are not valid for F_0.

Considering that $m_{k_C+1}(C\cup\{k\}) = m_{k_C}(C) \cup \{\min\{k, \underline{\gamma}(C \backslash m_{k_C}(C))\}\}$, it results $\sum_{j\in m_{k_C+1}(C\cup\{k\})} a_j \le \sum_{j\in m_{k_C}(C)} a_j + a_k \le \sum_{j\in m_{k_C}(C)} a_j + a_{\bar{\gamma}(J_0\backslash C)} \le b$ and, by Proposition 2, the inequality $X(C\cup\{k\}) \le k_C$ is not valid for F_0.

If $k_C = 1$, the inequality $X(C \backslash \{k'\}) \le k_C - 1$ is not valid for F_0. If $k_C > 1$ and it can be shown that $\sum_{j\in m_{k_C}(C\backslash\{k'\})} a_j \le b$, by Proposition 2 the inequality $X(C \backslash \{k'\}) \le k_C - 1$ will not be valid for F_0.

- If $k' \in m_{k_C}(C)$, it follows that $m_{k_C}(C \backslash \{k'\}) = m_{k_C+1}(C) \backslash \{k'\}$, hence $\sum_{j\in m_{k_C}(C\backslash\{k'\})} a_j = a_{\underline{\gamma}(C)} + \sum_{j\in m_{k_C}(C\backslash\{\underline{\gamma}(C)\})} a_j - a_{k'} \le \sum_{j\in m_{k_C}(C\backslash\{\underline{\gamma}(C)\})} a_j \le b$.

- If $k' \in C \backslash m_{k_C}(C)$, we have that $m_{k_C}(C \backslash \{k'\}) = m_{k_C}(C)$ and, consequently, $\sum_{j\in m_{k_C}(C\backslash\{k'\})} a_j \le \sum_{j\in m_{k_C}(C\backslash\{\underline{\gamma}(C)+\})} a_j \le b$.

The characterization given by Theorem 3 is not valid if $C = J_0$, since in this case $\bar{\gamma}(J_0 \backslash C)$ is not defined. Theorem 4 states a necessary and sufficient condition for J_0 to be a maximal cover from the set of covers implied by constraint (3).

Theorem 4. Let $X(J_0) \le k_0$ be the inequality induced by the cover J_0, where $k_0 = \max\{l \mid \sum_{j\in m_l(J_0)} a_j \le b\}$.

Then J_0 is a maximal cover from the set of covers implied by constraint (3) if and only if $\sum_{j\in m_{k_0}(J_0\backslash\{\underline{\gamma}(J_0)\})} a_j \le b$.

Proof. It follows from the proof of Theorem 3. (Note that J_0 is a non-trivial cover).

Example 2. Consider the 0-1 knapsack constraint

$$x_1 + 4x_2 + 4x_3 + 5x_4 + 7x_5 \leq 8 \quad (9)$$

Consider the cover $C_1 = \{2, 4, 5\}$ and let $X(C_1) \leq 1$ be its induced inequality. By Theorem 3, C_1 is a maximal cover from the set of covers implied by constraint (9), since $a_2 + a_3 = 8$.

Consider the cover $C_2 = \{3,4\}$ and let $X(C_2) \leq 1$ be its induced inequality. By Theorem 3, C_2 is not a maximal cover from the set of covers implied by constraint (9), since $a_3 + a_5 = 11$.

Consider the cover $C_3 = \{1, 2, 3, 4, 5\}$ and let $X(C_3) \leq 2$ be its induced inequality. By Theorem 4, C_3 is a maximal cover from the set of covers implied by constraint (9), since $a_2 + a_3 = 8$.

4. MINIMAL COVERS AND EXTENSIONS

Definition 6. A non-trivial cover C implied by constraint (3) such that $C^- = \phi$ is said to be **minimal** with respect to constraint (3) if $\sum_{j \in C \setminus \{k\}} a_j \leq b \quad \forall k \in C$.

Definition 7. Let C be a minimal cover with respect to constraint (3). The set $E(C) = C \cup \{j \in J_0 \mid j > \bar{\gamma}(C)\}$ is called the **extension** of C.

For the sake of completeness, let the well-known three propositions below.

Proposition 9. If C is a minimal cover with respect to constraint (3), then $k_C = |C| -1$ and $C \subseteq J_0$.

Proof. Let $k \in C$, $x_j = 1 \quad \forall j \in C \setminus \{k\}$ and $x_j = 0 \quad \forall j \in J \setminus (C \setminus \{k\})$. Since $(x_j)_{j \in J} \in F_0$, it must be $k_C = |C| -1$. Therefore, by Proposition 1 we have that $\sum_{j \in C} a_j > b$. Now, if $\exists\ k \in C \setminus J_0$ then $\sum_{j \in C \setminus \{k\}} a_j = \sum_{j \in C} a_j > b$, contradicting the fact that C is a minimal cover with respect to constraint (3).

Proposition 10. A set $C \subseteq J_0$ is a minimal cover with respect to constraint (3) with induced inequality $X(C) \leq |C| -1$ if and only if $\sum_{j \in C} a_j > b$ and $\sum_{j \in C \setminus \{\underline{\gamma}(C)\}} a_j \leq b$.

Proof. It follows from Propositions 1 and 9.

Proposition 11. If C is a minimal cover with respect to constraint (3), then

(1) E (C) is a non-trivial cover implied by constraint (3), and the inequality $X(E(C)) \leq |C| -1$ is induced by E(C).

(2) The inequality induced by C is dominated by the inequality $X(E(C)) \leq |C| -1$.

Proof. (1) It follows from Proposition 2, since $m_{|C|}(E(C)) = C$ and $\sum_{j \in C} a_j > b$.

(2) Trivial.

Theorem 5 shows that every maximal cover from the set of covers implied by constraint (3) is the extension of a unique minimal cover with respect to (3).

Accordingly, for identifying these maximal covers it suffices to identify the minimal covers with respect to (3), see Propositions 9 and 10. A characterization of those minimal covers whose extensions are maximal covers from the set of covers implied by constraint (3) is given in Theorem 6. Proposition 12 is required by the proof of Theorem 5.

Proposition 12. Let C and C' be two minimal covers with respect to constraint (3).

If $E(C) = E(C')$, then $C = C'$.

Proof. Without loss of generality, suppose that $\bar{\gamma}(C) \le \bar{\gamma}(C')$. If $\exists\ j \in C \setminus C'$, it follows that $j < \bar{\gamma}(C')$ and, so, $j \notin E(C')$, which contradicts the fact that $C \subseteq E(C) = E(C')$.

Consequently, it must be $C \subseteq C'$. Now, if $C \subset C'$, choosing $k \in C' \setminus C$ we have that $\sum_{j \in C' \setminus \{k\}} a_j \ge \sum_{j \in C} a_j > b$, whence C' is not a minimal cover with respect to constraint (3).

Theorem 5. If C is a maximal cover from the set of covers implied by constraint (3), then there exists a unique minimal cover with respect to constraint (3), say C', such that $E(C') = C$.

Proof. By Proposition 6, $C \subseteq J_0$ and its induced inequality is $X(C) \le k_C$, where $k_C = \max\{l \mid \sum_{j \in m_l(C)} a_j \le b\} < |C|$. Let $C' = m_{k_C+1}(C)$. Considering that $m_{k_C}(C \setminus \{\underline{\gamma}(C)\}) = C' \setminus \{\underline{\gamma}(C')\}$, by Theorems 3 and 4 it results $\sum_{j \in C' \setminus \{\underline{\gamma}(C')\}} a_j \le b$. Therefore, by Proposition 10 it follows that C' is a minimal cover with respect to constraint (3) and its induced inequality is $X(C') \le k_C$.

If $C = J_0$, it is clear that $E(C') = C$. If $C \subset J_0$, by the proof of Theorem 3 we have that $\bar{\gamma}(J_0 \setminus C) < \underline{\gamma}(C \setminus m_{k_C}(C))$ and, since $\underline{\gamma}(C \setminus m_{k_C}(C)) = \bar{\gamma}(C')$, it results $E(C') = C$.

Hence, by Proposition 12, C' is the unique minimal cover with respect to constraint (3) such that $E(C') = C$.

Theorem 6. Let C be a minimal cover with respect to constraint (3) and let $X(E(C)) \le |C| - 1$ be the inequality induced by E(C). Then E(C) is a maximal cover from the set of covers implied by constraint (3) if and only if one of the two following conditions is satisfied:

(1) $E(C) \subset J_0$ and $\sum_{j \in C \setminus \{\underline{\gamma}(C)\}} a_j + a_{\bar{\gamma}(J_0 \setminus E(C))} \le b$.

(2) $E(C) = J_0$.

Proof. It follows from Theorems 3 and 4. (Note that $\sum_{j \in C} a_j > b$ and $\sum_{j \in C \setminus \{\underline{\gamma}(C)\}} a_j \le b$, from which $\max\{l \mid \sum_{j \in m_l(E(C))} a_j \le b\} = |C|-1$ and $\sum_{j \in m_{|C|-1}(E(C) \setminus \{\gamma(E(C))\})} a_j \le b$).

Theorems 5 and 6 show that the extensions of the strong covers defined in Balas (1975) are the maximal covers from the set of covers implied by the related 0-1 knapsack constraint. So, this type of maximal covers can be used to obtain facets of the knapsack polytope, see e.g. [Balas (1975), Balas-Zemel (1978), Padberg (1975), Weismantel (1997) Wolsey (1976).

REMARK. In [2] it was shown that, given a 0-1 knapsack constraint, the inequality induced by the extension of a strong cover C is not dominated by any other inequality of the form $X(E(C')) \le |C'|-1$, where C' is a minimal cover with respect to the knapsack constraint and $|C'| = |C|$. It is worthy of note that we have shown that the inequality induced by the extension of a strong cover is not dominated by any other inequality of the form $X(C'^{+}) - X(C'^{-}) \le k_{C'} - |C'^{-}|$, where $C'^{+} \cup C'^{-}$ is a cover implied by the knapsack constraint.

Example 3. Consider the 0-1 knapsack constraint

$$2x_1 + 3x_2 + 3x_3 + 5x_4 + 6x_5 + 7x_6 + 7x_7 \le 12. \quad (10)$$

Let $C_1 = \{1, 4, 6\}$, $C_2 = \{2, 4, 5\}$ and $C_3 = \{1, 2, 3, 4\}$. By Proposition 10 it results that C_1, C_2 and C_3 are minimal covers with respect to (10); their extensions are $E(C_1) = \{1, 4, 6, 7\}$, $E(C_2) = \{2, 4, 5, 6, 7\}$ and $E(C_3) = \{1, 2, 3, 4, 5, 6, 7\}$.

Let (11), (12) and (13) be the inequalities induced by $E(C_1)$, $E(C_2)$ and $E(C_3)$ respectively.

$$x_1 + x_4 + x_6 + x_7 \le 2 \tag{11}$$

$$x_2 + x_4 + x_5 + x_6 + x_7 \le 2 \tag{12}$$

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 \le 3 \tag{13}$$

Then, by Theorem 6 it follows that $E(C_1)$ is not a maximal cover from the set of covers implied by constraint (10), but $E(C_2)$ and $E(C_3)$ are maximal covers.

Example 4. Consider the following 0-1 knapsack constraint taken from Example 2.1 in [15].

$$x_1 + x_2 + x_3 + x_4 + 3x_5 + 4x_6 \le 4 \tag{14}$$

It can be shown from Proposition 10 and Theorem 6 that the inequalities induced by the maximal covers from the set of covers implied by constraint (14) are (15)-(25).

$$x_1 + x_6 \le 1 \tag{15}$$

$$x_2 + x_6 \le 1 \tag{16}$$

$$x_3 + x_6 \le 1 \tag{17}$$

$$x_4 + x_6 \le 1 \tag{18}$$

$$x_5 + x_6 \le 1 \tag{19}$$

$$x_1 + x_2 + x_5 + x_6 \le 2 \tag{20}$$

$$x_1 + x_3 + x_5 + x_6 \le 2 \tag{21}$$

$$x_1 + x_4 + x_5 + x_6 \le 2 \tag{22}$$

$$x_2 + x_3 + x_5 + x_6 \le 2 \tag{23}$$

$$x_2 + x_4 + x_5 + x_6 \le 2 \tag{24}$$

$$x_3 + x_4 + x_5 + x_6 \le 2 \tag{25}$$

Thus, it can easily be verified that inequalities (26)-(30) are induced by covers implied by constraint (14), see inequalities (20) and (25).

$$x_1 + x_2 + x_3 + x_5 + x_6 \le 3 \tag{26}$$

$$x_1 + x_2 + x_4 + x_5 + x_6 \le 3 \tag{27}$$

$$x_1 + x_3 + x_4 + x_5 + x_6 \le 3 \tag{28}$$

$$x_2 + x_3 + x_4 + x_5 + x_6 \le 3 \tag{29}$$

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 \le 4 \tag{30}$$

Moreover, it can be shown that each of the inequalities (20)-(30) can be tightened by using coefficient increasing procedures, see [Escudero **et al**. (1998), Escudero-Muñoz (1998), Muñoz (1999)] among others. As a result, a complete inequality description of the polytope related to constraint (14) is obtained, see inequalities (1)-(16) from [Weismantel (1997)].

A procedure for identifying non-dominated inequalities induced by the extensions of certain minimal covers with respect to constraint (3) is presented in [Dietrich **et al**. (1993)], see also [Muñoz (1995)]. The two examples below taken from [Dietrich **et al**. (1993)] show that these extensions can be a very small fraction of the whole set of maximal covers from the set of covers implied by (3). This fact, together with the computational results that have been obtained in the literature by applying the algorithms given in [Dietrich et.al.(1993)] and by using the resulting covers to tighten a 0-1 model (see [Dietrich-Escudero(1993), Escudero **et al**. (1998)] among others), indicates that an approach for identifying maximal covers based on Proposition 10 and Theorem 6 could become very useful in 0-1 model tightening.

Example 5. Consider the 0-1 knapsack constraint

$$x_1 + 2x_2 + 3x_3 + 4x_4 + 5x_5 + 8x_6 + 9x_7 +$$

$$16x_8 + 17x_9 + 32x_{10} + 33x_{11} + 64x_{12} + 65x_{13} \leq 128. \qquad (31)$$

It can be shown from Proposition 10 and Theorem 6 that there are 83 maximal covers from the set of covers implied by constraint (31), but only 3 of them are identified by the procedure proposed in Dietrich **et al**. (1993).

Example 6. Consider the 0-1 knapsack constraint

$$5x_1 + 6x_2 + 6x_3 + 7x_4 + 9x_5 + 10x_6 + 10x_7 +$$

$$11x_8 + 12x_9 + 14x_{10} + 14x_{11} + 15x_{12} + 15x_{13} + 16x_{14} +$$

$$17x_{15} + 18x_{16} + 20x_{17} + 21x_{18} + 22x_{19} + 22x_{20} + 23x_{21} \leq 119. \qquad (32)$$

It can be shown from Proposition 10 and Theorem 6 that there are 23514 maximal covers from the set of covers implied by constraint (32), but only 41 of them are identified by the procedure proposed in Dietrich **et al**. (1993).

5. CONCLUSIONS

In this paper we have introduced the concept of maximal covers from a set of covers and we have given some characterizations of several types of covers. It has been shown that the maximal covers from the set of covers implied by a 0-1 knapsack constraint are the extensions of the so-called strong covers. Some situations that illustrate the benefits of maximal cover identification in 0-1 model tightening have also been shown.

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