A STATISTICAL EQUILIBRIUM APPROACH FOR PREFERENCE AGGREGATION

Gustavo Mazcorro-Téllez[1], Postgraduate Studies Division, Faculty of Engineering, UNAM & UPIICSA, National Polytechnic Institute. Mexico City

ABSTRACT
This paper is designed to expose a conceptual framework to analyse the major problem of preference aggregation. The proposal is a theoretical approach based on Statistical Mechanics and Social Choice Theory. From the approach, a specific procedure for preference aggregation is proposed in accordance with two main criteria: optimality and stability. This, in regard to axiomatic demands imposed to the ranking of preferences that is sought. Thus far, the framework differs with respect to most of the approaches for preference aggregation, such as those involved in voting theory. In the proposal, optimality is incorporated as a minimal-variability restriction, and stability makes use of equilibrium conditions coming from statistical mechanics. For the discussion, a specific measurement procedure for individual-ranking- evaluation is suggested, and the collective choice is the ranking that satisfies the two referred criteria at a better extent.

Key words: preference aggregation, statistical equilibrium, weak order, ranking evaluation, social choice.

RESUMEN
Se presenta una estructura conceptual para el análisis del problema más amplio de agregación de preferencias. La propuesta es una aproximación teórica soportada en conceptos de Mecánica Estadística y la Teoría de Elección Social. De la aproximación se deriva un procedimiento específico de agregación preferencial bajo dos criterios fundamentales: Optimalidad y Estabilidad. Esto, de acuerdo a condiciones axiomáticas impuestas al *ranking* final. En este sentido, la estructura conceptual difiere con respecto a la mayoría de las aproximaciones utilizadas en agregación de preferencias, tales como aquellas que involucra la Teoría de Votaciones. En la propuesta la optimalidad se incorpora como una restricción de variación mínima y la estabilidad se propone en términos de condiciones de equilibrio que provienen de la Mecánica Estadística Clásica. Para la discusión se sugiere un procedimiento específico de evaluación de *rankings* y el ranking final es aquel que mejor satisface los dos criterios señalados.

Palabras clave: agregación de preferencias, evaluación de rankings, estabilidad y optimalidad.

1. INTRODUCTION

The study of the procedures for collective decision-making has predominantly focussed on methods that propose the construction of group or social preferences starting from individual-preference statements. The basic assumption of this domain of problems is that, given a set of alternatives, when a decision-making group is engaged in the purpose of choosing one of them, the preference pattern of the participants must be combined via some rational procedure. So, in these terms, a collective decision-making problem can be considered as a preference aggregation process. Situations of this sort compose the object of the present work; moreover its main contribution is a theoretical approach to analyse the conditions of the preferences expressed by a decision-making group for the purpose of establishing a collective hierarchy –or group preference– on a given set of alternatives.

Particular denominations are useful to designate a preference aggregation procedure. For instance, Campbell (1976) utilizes the term ‘constitution’ in his dissertation about democratic preference functions. But, in point of fact, democracy, justice and fairness, constitute the main guidelines for collective decision-making, especially when a normative standpoint is assumed. To this respect, it is pertinent to highlight that most of the studies in the field of collective decision-making have not considered, in a fundamental way, the use of approaches such as optimality or stability, neither in the study of collective decision-making, nor in the development of procedures for preference aggregation. Moreover, it is observed that such scarcity contrasts with a number of Decision Analysis’ proposals, in which it is prescribed to questioning the ‘stability’ of a

[1]gustavom@antares.upiicsa.ipn.mx

solution by means of any type of sensibility evaluation [e.g. Keeney and Raiffa (1976), p.100; Hogarth (1988), p.182].

From this idea on, in this paper a conceptual structure besides a specific procedure for preference aggregation are exposed. The proposal comprises a theoretical appraisal based on statistical arguments; hence, it is fairly related to the generality of the studies in this field of research, which basically have been concentrated on analysing a variety of procedures for preference aggregation most of them based on some kind of majority-rule approach [e.g. Arrow and Raynaud (1986), Fishburn (1971,1973), French (1988)]. To know, the proposal in this paper evocates nearly axiomatic conditions that a ranking of alternatives must accomplish to be considered 'the best', and consequently may be acknowledged as the group ranking. The specific conditions that are imposed to that solution are optimality and stability. To introduce the argument, a basic schedule for preference evaluation is outlined, which is followed by a specific method of aggregation. The approach regards to a statistical equilibrium concept picked out from Classic Statistical Mechanics. The discussion concludes with the analysis of a hypothetical industrial situation that involves a decision-making committee. It is considered that in the resolution of that case the convenience of the proposed method becomes patent.

2. RECURRENT PROFILES AND SETS OF SOLUTIONS

One of the problems that has engaged more investigation in collective decision-making concerns the appearance of cycles in final rankings of group preference. In this paper, reference is made to papers by Arrow and Raynaud (1986), Fishburn (1973), French (1988) and Meyer and Brown (1998), in which important conclusions on this matter are exposed. Cyclic or recurrent rankings of group preference appear when specific methods of preference aggregation are applied in most of the cases via some outline of majority rule, such as Borda count or Condorcet's winner. The problem resides on cyclic or recurrent-group-rankings denote inconsistencies –intransitivities– from the point of view of the theory of preferences.

In this paper, a *recurrent profile of individual preferences* is defined to be a triple, such as

$$v_1(\{a,b,c\}) = a,b,c$$

$$v_2(\{a,b,c\}) = b,c,a$$

$$v_1(\{a,b,c\}) = c,a,b$$

Where, $v_i(V) = v_i(\{a,b,c\})$, i = 1,2,3 indicates that individual i, when facing alternatives a, b and c, specifies a ranking of preferences as the one given by the expression on the right. Here, the term 'individual' refers to any agent that is able to manifest preferences. Individuals will be called indistinctly agents, voters or decision-makers.

In connection with recurrent profiles, one of the most disturbing results occurs when a simple majority rule –Condorcet's winner– is applied and the definition of the preference-group-hierarchy depends on the way this rule is performed (French; p. 281). The problem focuses on such hierarchy depends on the order of application of the majority rule, instead of the information comprised in the individual preference profile. Certainly, if the existence of a *better-off* hierarchy of alternatives is assumed to exist, then it must be rather exigible that such hierarchy shall not be defined by the method, but the conditions that the specific profile comprises. Therefore, it is constructive to have a rational approach that allows 'evaluating' the 'adequateness' of each possible solution. In any case, the information provided by the profile is the diversity of preferences, and the adequateness of the approach is evaluated in terms of the points expressed along the discussion in next sections.

On the other hand, whenever a profile of preferences is admitted to be valid, then it must be also admitted a significant probability that any solution, as prescribed or determined through any aggregation procedure, will not satisfy entirely all the intervening single-decision-makers. Further, it is rather probable that majority support stands for being the best appraisal to establishing a solution for the problem, even though in many cases it is only possible to establish aggregation procedures which are 'acceptable', instead of being optimal in any sense. Along with this, what has been considered to be a solution for collective decision-making problems has been the definition of procedures with the capacity to create assent among the participants; yet, not regarding the solution, but the method which leads to it.

In connection with aforementioned statements, the proposal that has supported most of the studies in collective decision-making –or social choice– is Arrow's axiomatic system. This model is composed by a set of axioms, which are proposed to accomplish the requirements of any procedure for preference aggregation. In Arrow's axiomatic construction, aggregation procedures are demanded to assimilate the most elementary ideas of justice, fairness and democracy (Arrow and Raynaud, p.18).

2.1 Sets of solutions

In this section, it is presented an approach that combines some results arising from the Theory of Preferences and the major model of Classic Statistical Mechanics. As it has been asserted, one of the goals of this paper is to integrate an analytic framework for studying the preference aggregation procedures. The discussion is adjusted to the following stance and notations.

Let $V = \{v_1, v_2, ..., v_n\}$ be a set of single decision-makers. It is assumed that each element in *V* has at its disposal any of the M states composing the set $S^m = \{\psi_1^m, \psi_2^m, ...\psi_M^m\}$, which will be called *solution set of order* m. Elements $\psi_j^m \in S^m$ j = 1,2,..,M will represent exhaustively the orderings –permutations– that can be built with the m elements of a –basic– set of alternatives $A_m = \{a_1, a_2, ..., a_m\}$. Explicitly, a state $\psi_j^m \in S^m$ stands for a structure of weak preference in the way $\psi_j^m = a_{(1)} \succsim a_{(2)} \succsim \cdots \succsim a_{(m)}$, where $a_{(k)} \in A_m$; k = 1,...,m.

In accordance with the language, as stated so far, reference to the previous structure will be made by using terms as hierarchy or ranking. Indeed, a weak preference corresponds to a preference binary relation, which satisfies being transitive and asymmetric [French, p.283; Fishburn (1973), p.75].

Now, a specific arrangement regarding S^3 is considered, in accordance with the next outline.

2.2 The space Ψ^3 and higher order spaces

For the case of three alternatives, without looseness of generality, let $A_3 = \{a, b, c\}$, so m = 3 and M = m! = 6. The M possible rankings of the elements in A_3 are listed below in such a way that these, respectively, compose recurrent profiles of individual preference. This is,

$$\begin{array}{ll} \psi_1^3 = a \succsim b \succsim c & \psi_4^3 = c \succsim b \succsim a \\ \psi_2^3 = b \succsim c \succsim a & \psi_5^3 = a \succsim c \succsim b \\ \psi_3^3 = c \succsim a \succsim b & \psi_6^3 = b \succsim a \succsim c \end{array}$$

$(\psi_1^3, \psi_2^3, ..., \psi_3^3)$ will be called *third-order space of solutions* and designated by Ψ^3, which is merely an ordered structure containing the elements in S^3. For the cases where $|A| \geq 3$, the procedure for constructing the respective spaces of solutions is as follows. For m = 4, $A_4 = \{a, b, c, d\}$ and M = 24. Then, starting from the space Ψ^3 the proposed structure is:

$$\begin{array}{llll} \psi_1^4 = d \succsim a \succsim b \succsim c & \psi_7^4 = a \succsim d \succsim b \succsim c & \psi_{13}^4 = a \succsim b \succsim d \succsim c & \psi_{19}^4 = a \succsim b \succsim c \succsim d \\ \psi_2^4 = d \succsim b \succsim c \succsim a & \psi_8^4 = b \succsim d \succsim c \succsim a & \psi_{14}^4 = b \succsim c \succsim d \succsim a & \psi_{20}^4 = b \succsim c \succsim a \succsim d \\ \psi_3^4 = d \succsim c \succsim a \succsim b & \psi_9^4 = c \succsim d \succsim a \succsim b & \psi_{15}^4 = c \succsim a \succsim d \succsim b & \psi_{21}^4 = c \succsim a \succsim b \succsim d \\ \psi_4^4 = d \succsim c \succsim b \succsim a & \psi_{10}^4 = c \succsim d \succsim b \succsim a & \psi_{16}^4 = c \succsim b \succsim d \succsim a & \psi_{22}^4 = c \succsim b \succsim a \succsim d \\ \psi_5^4 = d \succsim a \succsim c \succsim b & \psi_{11}^4 = a \succsim d \succsim c \succsim b & \psi_{17}^4 = a \succsim c \succsim d \succsim b & \psi_{23}^4 = a \succsim c \succsim b \succsim d \\ \psi_6^4 = d \succsim b \succsim a \succsim c & \psi_{12}^4 = b \succsim d \succsim a \succsim c & \psi_{18}^4 = b \succsim a \succsim d \succsim c & \psi_{24}^4 = b \succsim a \succsim c \succsim d \end{array}$$

Observe that such structure is got by introducing *d* as the first –or more preferred– alternative for the first six states of the space; as the second preference for the following six, and so on, until *d* is situated as the less favourite alternative and consequently it is located lastly. Evidently, whenever a decision-maker experiences the exercise of manifesting its preferences regarding a specific space of solutions, this will prefer the state that fully matches its preferences; and he will manifest some hierarchy of preferences amongst the states in the space.

For such cases in which a bigger number of alternatives is concerned, the construction of spaces of solutions will be carried out in a similar way. This is, to compose a space of solutions of order *m+1* the arrangement corresponding to the space of solutions of order *m* shall be taken to locate the new alternative in first term for the first *m*! elements, in second place for the following m!, and so forth. It will be important to remember that the classifications which defines the spaces of solutions of higher order than three shall be gotten starting from the space of solutions of the current order less one. Certainly, the structure of a space of solutions as defined above is not assumed to have inherent theoretical properties, aside of incorporating recurrent profiles into a well-ordered and rather tractable frame that comprises all the possible rankings from a set of alternatives. (Present examination concentrates on the study of solution-spaces of 6^{th} order or lesser, however, the concepts under discussion are proposed to be valid for spaces of higher order.)

3. MEASUREMENT APPROACHES

The approach for preference measurement that is suggested consists on observing the number of differing binary preferences among a certain profile and the elements in the corresponding space of solutions. It is admitted that each state ψ_j^m defines for a voter i –generic–, a specific state of 'agreement' in accordance with voter's hierarchy of preferences on the set A_m. Consequently, for any decision-maker the state that maximizes his satisfaction will be the one which utterly agrees with respect to his preferences, and the state of minimum satisfaction will be the one showing total disagreement. For instance, a decision-maker whose preferences on the set A_5 = {a, b, c, d, e, f} match the hierarchy $e \succsim a \succsim b \succsim f \succsim c \succsim d$ (individual order of preferences), when facing the group ranking $a \succsim_g b \succsim_g f \succsim_g c \succsim_g d \succsim_g e$ (group order of preferences defined by some elective process), this will differ in five relative positions. Specifically, the decision-maker will be in disagreement with regard to the positions that keep the group preferences $a \succsim_g e$; $b \succsim_g e$; $f \succsim_g e$; $c \succsim_g e$ y $d \succsim_g e$ (preferences in disagreement) because his are contrary. It is important to clarify that it is not argued that such outline for evaluation stands, neither for the lone one possible, nor the most representative for any decision-maker. However, when assuming an individual decisive behaviour as represented by a weak order, then it can be assured those disagreements exist. For a better understanding of these ideas, in Figure 1 an evaluation of the number of preferences in disagreements for a sole decision-maker that faces Ψ^3 is presented.

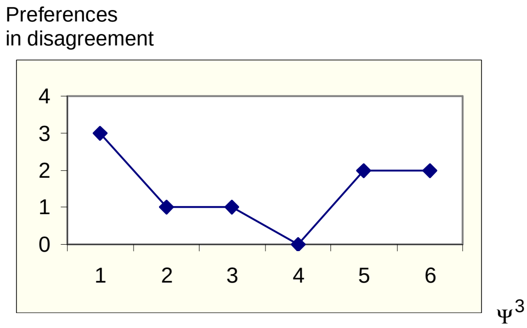

Figure 1. Quantification of the preferences in disagreement for an individual dealing with Ψ^3 and whose order of preferences matches $\psi_4^3 = c \succsim b \succsim a$.

Some arguments are offered to support the proposal for measurement. Intuitively, it is assumed that the satisfaction pertaining to a particular agent can be evaluated in accordance with the extent that a specific ranking matches his own; certainly the one this agent would propose as the solution for the collective decision-making process. On this matter, reference is done to the axiomatic system proposed by Masarani and Gokturk (1989) about the *fair matching problem*.

In terms as used by Masarani and Gokturk, the fair matching problem assumes the existence of two groups of agents, these represented by *X*, *Y*. Each group counts *n* members and the possibility of conforming "fair" matches with the members of *X*, *Y* is questioned. It is assumed that, as a result of the preferences of the elements within sets X and *Y*, each individual possesses a hierarchy of preferences on the other group's individuals. Those classifications rank the most preferred individual in first place, and so on, till locating the less preferred in last place. Along with Masarani and Gokturk, the satisfaction of any individual may be evaluated by knowing how near to his first preference is the individual with which this individual is matched.

On one hand, the proposal by the referred authors consists on designating an external agent to perform as a 'referee' to achieve the matching, and the problem is constrained by provisioning that the solution arising from the referee shall be accepted unrestrainedly. External agent's capacity –any procedure– is analysed accordingly with two fundamental questions: (1) what does constitute a fair matching? and (2) for any structure of preferences, does such a procedure of fair matching exist? This position leads those authors to propose an axiomatic scheme, whose verification is used to guarantee ‘fairness’ on referee's performance in the application of the matching algorithm. This approach is based upon Rawls’ concept of justice.

Indeed, the resource of a referee is not unaware to the theoretical structure of preferences. In particular, French discusses the nature and performance of this sort of agents “supra-decision-makers” (French, p. 298) in the case of negotiation problems. In French's conclusion, the effectiveness of a referee depends mainly on its proceeding remains attached to specific conditions. A similar idea is orchestrated here.

Two of the axioms proposed by Masarani and Gokturk are judged as useful to argue the tender of the present paper. The first one refers to Maximin Optimality, which means that a matching algorithm must locate individuals pertaining to each set in the same level of satisfaction. This is to say that none of the groups may be favoured. Moreover, it is proposed to maximize the correspondence in terms of distance minimizing. A second axiom is proposed as an approach for ‘stability’; in this case, it is simply asked the algorithm to generate ‘stable’ matches. Particular interpretations of maximin optimality and stability conditions will be used in what follows.

4. FUNCTION OF AGREEMENT AND EQUILIBRIUM CONDITIONS

As exposed so far, space Ψ^m possesses size $|\Psi^m| = m! \equiv M$ and consequently the number of disagreements that a decision-maker may exhibit with regard to states in Ψ^m varies from 0 to $C_2^m = \binom{m}{2} \equiv K$.

That is, variations are constrained to belong to the integer set {0,1,...,*K*}. For instance, in the case shown in Figure 1 it can be observed *0*,*1*,*2* or *3* disagreements; last value corresponding evidently to the result of $C_2^3 = \binom{3}{2} = \frac{3!}{2!(3-2)!} = 3$. States of disagreement will be represented by E_k, k = 1,...,K, where

$$\begin{array}{ccc} 0 & \rightarrow & E_0 \\ 1 & \rightarrow & E_1 \\ . & . & . \\ . & . & . \\ . & . & . \\ K & \rightarrow & E_k \end{array}$$

This representation defines the set $E = \{E_0, E_1, ..., E_{K-1}, E_K\}$, which will be called *energetic values set of the space of solutions*. These values are assumed to be equal to the integer numbers in the column on the left. To say, $E_0 = 0$, $E_1 = 1$, . . . , $E_k = K$. Such designation comes from the intuitive appealing that any individual, when being offered or imposed any state ψ_j^m differing from his, would intend to accomplish the necessary relative movements to bring it to match his own. In other words, unmatched rankings come off disagreement in decision-makers and cause the main system to become unstable.

4.1 Optimality conditions

Let V = {$v_1, v_2, ..., v_n$} be a set of decision-makers participating in a collective decision-making process in which the set of alternatives $A_m = \{a_1, a_2, ..., a_m\}$ is evaluated. Now, define the family of functions $\Theta = (\theta_1, \theta_2, ..., \theta_n)$, where θ_i is a function related to decision-maker *i* = 1,...,*n*, and whose effect is to evaluate the number of disagreements in i, re elements in Ψ^m .

Symbolically,

$\theta_i : \Psi^m \mapsto E$, for *i* = 1,...,n.

Therefore, when decision-maker i evaluates $\psi_j^m \in \Psi^m$, he associates an energetic value $e_i^j \in E$. This is,

$\theta_i(\psi_j^m) = e_i^j$

Each of $\theta_i \in \Theta$ will be called an *individual agreement function* and the family Θ will be called *collective agreement*.

Figure 8 shows the graph of one of the many possible individual agreement functions. In that case, decision-maker faces $A_6 = \{a, b, c, d, e, f\}$ and his ranking of preferences matches $a \succsim b \succsim c \succsim d \succsim e \succsim f$, which corresponds to $\psi_{715}^{6} \in \Psi^{6}$.

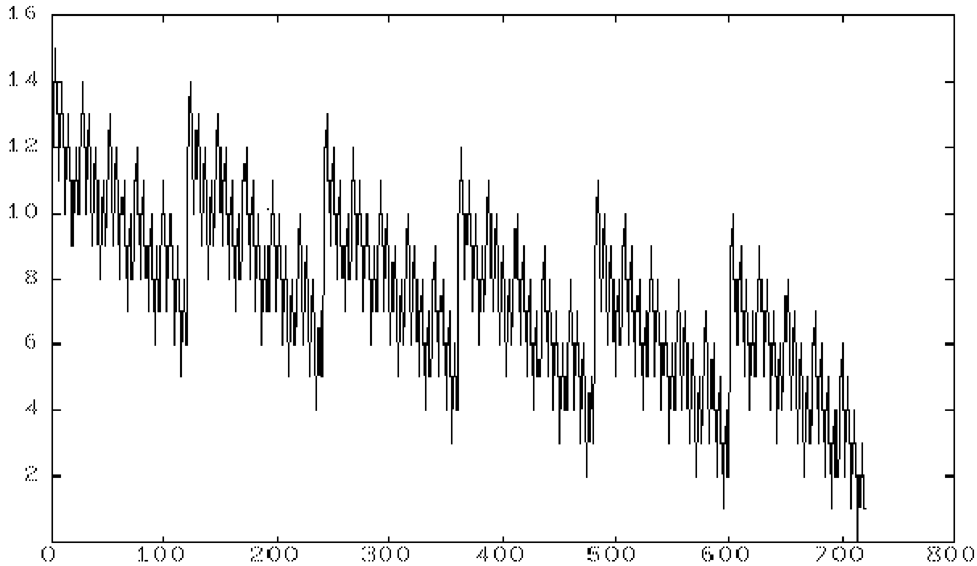

Figure 2. Individual agreement function pertaining to a decision-maker 'type' $\psi_{715}^{6} \in \Psi^{6}$.

Now, suppose that as a result of some aggregation procedure a solution represented by $\psi_j^m \in \Psi^m$ has been admitted. Then, applying the function

$$\Theta(\psi_j^m) = \left(\theta_1(\psi_j^m), \theta_2(\psi_j^m), \ldots, \theta_n(\psi_j^m)\right) = \left(e_1^j, e_2^j, \ldots, e_n^j\right)$$

a collective agreement is obtained in terms of a n-uple of energetic values for the *n* participants. If now a *function of aggregation* is defined as

$$\Im(\Theta(\psi_j^m)) = \sum_{i=1}^{n} e_i^j = e_1^j + e_2^j + \cdots + e_n^j \equiv \Sigma_j . \qquad \ldots(1)$$

Then, under this measurement, state ψ_j^m will represent an optimal solution to the problem whenever there is not a state $\psi_l^m \in \Psi^m$, so that $\Sigma_l < \Sigma_j$.

It is also observed that the best value for equation (1) corresponds to zero; yet this value equals zero under conditions of unanimity. Only in that case there are not discrepancies and the optimal solution corresponds to a value of the function of aggregation equals zero. Moreover, due to the fact that expression (1) does not constrain any configuration of energetic values, then it can be asserted that function (1) is in the aim of first Arrow's axiom; this is, voters express freely their preferences amongst all the states in the space of solutions [e.g. Arrow and Raynaud (1986)].

4.2 Equilibrium Conditions

In this section, a statistical model for analysing stability conditions of a solution $\psi_j^m \in \Psi^m$ is presented. The pattern is mainly adjusted to the structure that utilizes Classic Statistical Mechanics for the treatment of a system of *n* identical particles [Alonso and Finn (1980), Armstrong and King (1970), Reif (1969)]. Indeed, present paper adopts a position similar to Jaynes (1957), who proposes Statistical Mechanics as a theoretical approach not necessarily standing or physical-phenomena- representative. In Jaynes' work, applications for Statistical Mechanics coming from information theory are used, which carry out estimates of probability densities on the basis of partial knowledge. These are denominated maximum entropy estimates.

Assume that a preference aggregation procedure leads to the solution $\psi_j^m \in \Psi^m$. Next, consider the sequence $n_0, n_1, \cdots, n_K$, whose values correspond to the number of voters that given ψ_j^m prefer the state as denoted by sub-indexes. In words, n_0 represents the number of voters in state E_0; n_1 those in state E_1 and this way, successively, until considering n_k representing the number of voters in state E_k. Distribution $n_0, n_1, ..., n_k$ will be called a partition of n, therefore $n = n_0 + n_1 + \cdots + n_K = \sum n_i$ and this quantity is assumed to remain constant.

Further, reference to the following statistical postulate is made: "The probability of a certain partition is proportional to the number of different ways in which particles can be distributed among the available energy states to produce the partition (Alonso and Finn; p.449). In such a case, the probability of partition $n_0, n_1, \cdots, n_K$ is proportional to $\frac{n!}{n_0! n_1! \cdots n_K!}$. The underlying interpretation is 'particles'≈'voters', and energy states corresponding to energetic values.

In accordance with statistical mechanics statements, a fundamental condition for ψ_j^m to be an equilibrium solution is that partition $n_0, n_1, \cdots, n_K$ corresponds to an equilibrium state. For clarity, it is now appealed to the concepts exposed by Reif (1969, p. 120). "If an isolated system reaches with the same probability each of its accessible states, then it is in equilibrium." Likewise, "If an isolated system does not reach with the same probability each of its accessible states, it is not in equilibrium. It spreads then to vary with time until it reaches that equilibrium situation in which it meets finally with the same probability each one of their accessible states." (Reif, p.121). Lastly, "Whenever a system is in equilibrium, it reaches with the same probability anyone of its accessible states" (Reif; p.121).

Previous conditions demand allowance for the system to take any $\psi_j^m \in \Psi^m$ as a solution. Naturally, that is kindred to the demands of unrestrained domain that expresses first Arrow's axiom, under the interpretation that any hierarchy within the considered space of solutions must be admitted as the group hierarchy. If the conditions of unrestrained domain are applied to the individual preference hierarchies, then (1) determines a density of probabilities in which probability values associated with energetic values depend exclusively on the space of solutions.

It can be observed –Figure 1– that the number of possibilities leading to energy values is different for different values of *m*. In Chart 1 relative numbers of possibilities are presented, which drive to energy states in $E = \{E_0, E_1, ..., E_{k-1}, E_k\}$ for spaces of solutions in the range $3 \le m \le 6$.

It is considered superfluous to include the derivation of this result. (For it, see e.g. Alonso and Finn, p. 249-250.)

Ψ^3, k = 3		**Ψ^4, k = 6**		**Ψ^5, k = 10**		**Ψ^6, k = 15**	
E_i	g_i	E_i	g_i	E_i	g_i	E_i	g_i
0	1/6	0	1/24	0	1/120	0	21/720
1	2/6	1	3/24	1	4/120	1	5/720
2	2/6	2	5/24	2	9/120	2	14/720
3	1/6	3	6/24	3	15/120	3	29/720
		4	5/24	4	20/120	4	49/720
		5	3/24	5	22/120	5	71/720
		6	1/24	6	20/120	6	90/720
				7	15/120	7	101/720
				9	4/120	9	90/720
				10	1/120	10	71/720
						11	49/720
						12	29/720
						13	14/720
						14	5/720
						15	1/720

Chart 1. Probabilities of obtaining different energy values, assuming unrestrained participation of voter

and different spaces of solutions.

In Chart 1, g_i's denote probabilities for different energy values assuming respective values of E under unrestrained participation of voters. Observe that those values depend exclusively on the space of solutions, so they can be determined by means of any function of individual agreement. Values presented in Chart 1 are also shown in Figure 3; there it can be noted that some 'normality' is accomplished.

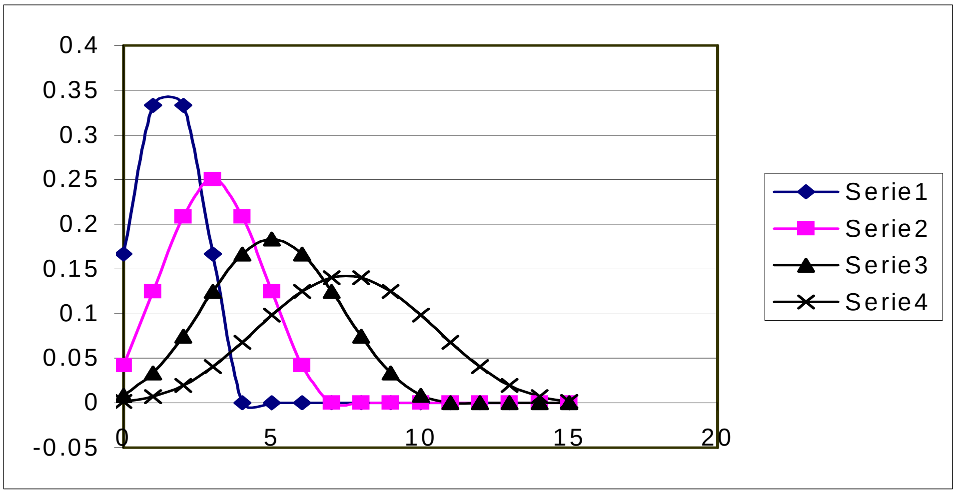

Figure 3. Graph forms of the values presented in Chart 1.
The series correspond to values m = 3, 4, 5, and 6, respectively.

In consequence, the density of probabilities corresponding to partition $n_0, n_1, \cdots, n_k$, is given by

$$f(n_0, n_1, ..., n_K) = \frac{g_0^{n_1} g_1^{n_2} \cdots g_K^{n_K}}{n_0! n_1! \cdots n_K!} = \prod_i \frac{g_i^{n_i}}{n_i!} \qquad ... (2)$$

and it vanishes in any other site.

Expression (2) corresponds to the probability of a distribution in Maxwell-Bolzman's Statistical Mechanics [Alonso and Finn; p. 451]. Further, consider the equality

$$U = n_0 E_0 + n_1 E_1 + \cdots + n_K E_K \qquad ... (3)$$

whose terms will be designated *total energy* and *total energetic configuration of the system*, respectively. Now, since ψ_j^m has been supposed as the solution of the process, then it is possible to equal the expressions (1) and (3), that correspond to the total energy of the system and the function of aggregation, respectively. This is,

$$U = \Im(\Theta(\psi_j^m)) \equiv \Sigma_j$$

$$n_0 E_0 + n_1 E_1 + \cdots + n_K E_K = e_1^j + e_2^j + \cdots + e_n^j$$

Here it is important to emphasise that term on the left side of the last equality corresponds to an energetic configuration, but the term on the right is the result of applying the aggregation function to the state ψ_j^m, which has been conceived as the solution of the process.

To obtain the basic equilibrium conditions in statistical mechanics the most probable distribution must be sought for the values $n_0, n_1, \cdots, n_K$ and total energy *U*. By definition, this result is got by determining the maximum of the values of $f(n_0, n_1, ..., n_K)$, simply due to the maximum of *f* is its more probable value.

For the goals of this paper, it is not considered useful to be plentiful in the mathematical construction through which the expressions of maximum probability are derived (reader may verify this upshot in Alonso and Finn, pp. 452-453). The result that one obtains, regarding the partition of maximum probability of an energy value U, is

$$n_i = g_i \exp[-\alpha - \beta E_i] \quad i = 0,\dots,K \qquad \text{...(4)}$$

With α and β, quantities representing *properties* of the system.

When expressing α as a function of the total number of voters *n*, expression (4) yields

$$\begin{aligned} n &= n_0 + n_1 + \cdots + n_K = g_0 \exp(-\alpha - \beta E_0) + \cdots + g_K \exp(-\alpha - \beta E_K) \\ &= \exp(-\alpha)[(g_0 \exp(-\beta E_0) + \cdots + g_\kappa \exp(-\beta E_K)] \\ &= \exp(-\alpha)\left(\sum_i g_i \exp(-\beta E_i\right) = \exp(-\alpha) Z \end{aligned}$$

where

$$Z \equiv \sum_i g_i \exp(-\beta E_i) \qquad \text{... (5)}$$

Expression (5) is called a *partition function*. If additionally, it is written

$$\exp(-\alpha) = n/Z$$

then expression (4) becomes

$$n_i = \frac{n}{Z} g_i \exp[-\beta E_i] \quad i = 0,\dots,K \qquad \text{...(6)}$$

This is the so-called Maxwell-Boltzman's law of distribution (Armstrong and King, p. 408; Alonso and Finn, p. 451).

Now, given a partition $n_0, n_1, \cdots, n_K$, its average energy value can be calculated appealing to the usual definition of average value in Probability Theory. Specifically,

$$\bar{E} = \frac{1}{n}\left[\sum_i n_i E_i\right] \qquad \text{...(7)}$$

And, for the distribution of agreement equilibrium with equation (6), one has,

$$\bar{E} = \frac{1}{Z}\left[\sum_i g_i E_i \exp(-\beta E_i)\right] \qquad \text{...(8)}$$

Lastly, if the system is in equilibrium, so it can be appealed to the equality $U = n\bar{E}$, and substituting it in (8) yields

$$U = n\bar{E} = \frac{n}{Z}\left[\sum_i g_i E_i \exp(-\beta E_i)\right] \qquad (9)$$

Equation (9) characterizes an 'equilibrium' among the variables involved in the system. Moreover, this equation allows finding β as a function of U, or equivalently of a Σ_j as found by applying some aggregation procedure, such as the one equation (1) bears. Hence, stability conditions in a collective decision process are therefore certain according to whether the configuration of voters' preferences, given a solution, corresponds to a statistical equilibrium configuration.

The proposed concepts are now used to solve the following case.

5. A CASE OF STUDY

Bring to mind a state of affairs in which a company confronts an extremely critical situation, so that, to solve the problem a decision-making committee has been integrated. Let $V=\{v_1, v_2, v_3, v_4, v_5\}$ be a set that groups the committee's members, these are

v_1 : General Manager.
v_2 : Trade-union representative.
v_3 : Personnel Manager.
v_4 : Production Manager.
v_5 : Government Representative.

A set of m = 6 alternatives is considered in accordance with the following assertions

a: To fire some personnel
b: To reduce salaries
c: To diminish daily-labour extent (and consequently the salaries)
d: Not to carry out any action.
e: To request government support.
f: To sell the company or to look for a society.

Now, preference orders of the committee members compose the profile:

$$v_1(\{a, b, c, d, e, f\}) = a \succsim b \succsim c \succsim d \succsim e \succsim f$$
$$v_2(\{a, b, c, d, e, f\}) = b \succsim c \succsim e \succsim d \succsim f \succsim a$$
$$v_3(\{a, b, c, d, e, f\}) = e \succsim a \succsim b \succsim f \succsim c \succsim d$$
$$v_4(\{a, b, c, d, e, f\}) = a \succsim b \succsim f \succsim d \succsim e \succsim c$$
$$v_5(\{a, b, c, d, e, f\}) = b \succsim f \succsim d \succsim c \succsim a \succsim e$$

So, m = 6 and the space of solutions under concern is Ψ^6, whose size is M = 6! = 720. Hence, energetic values and probabilistic assignments correspond to values within column $\Psi^6, k = 15$ in Chart 1.

To solve the problem, function (1) must be applied to each state in Ψ^6. Subsequently, it is necessary to identify the state which corresponds to the minimum of $j = 1,\ldots,720$. The state ψ_j^6 that is obtained in this way might not be unique; however, any state corresponding to the minimum value is considered to be an equivalent solution, in accordance with the *total energy* and the *total energetic configuration of the system*. In Figure 10, a complete description of the aggregation function on Ψ^6 is presented in terms of the profile of preferences. To obtain the results in that graph, numeric routines were used through PC-Matlab language [Moler **et al** (1984a, 1984b)].

The minimum value for Σ_j obtained by means of this procedure is 24, which corresponds to the state

$$\psi_{355}^6 = a \succsim_g b \succsim_g f \succsim_g c \succsim_g d \succsim_g e,$$

Where $\succsim_g$ denotes group preference.

On the other hand, the maximum value is 51, which corresponds to the state

$$\psi_{364}^6 = e \succsim_g d \succsim_g c \succsim_g f \succsim_g b \succsim_g a$$

The group agreement for ψ_{355}^6 is

$$\Theta(\psi_j^6) = \left(\theta_1(\psi_j^6), \theta_2(\psi_j^6), \ldots, \theta_n(\psi_j^6)\right) = \left(e_1^j, e_2^j, \ldots, e_n^j\right)$$

so

$$\Theta(\psi_{355}^{6})=(\theta_1(\psi_{355}^{6}),\theta_2(\psi_{355}^{6}),\theta_3(\psi_{355}^{6}),\theta_4(\psi_{355}^{6}),\theta_5(\psi_{355}^{6}))=(3,9,5,2,5)$$

In words, first individual grants energetic value 1 to ψ_{355}^{6}; second individual grants 2, and so on. State ψ_{355}^{6} is recognized as the best solution because there is not a state providing a better value than 24. It is interesting to notice that the solution obtained in this example by using Condorcet's Method [e.g. Fishburn (1971), French, p. 280, Meyer and Brown (1998)] is an intransitive profile. Indeed, this method leads to the establishment of the following preferences:

$a \succ b$	$b \succ c$	$c \succ d$	$d \succ e$	$e \succ f$
$a \succ c$	$b \succ d$	$c \succ e$	$d \prec f$ *	
$a \succ d$	$b \succ e$	$c \prec f$ *		
$a \succ e$	$b \succ f$			
$a \succ f$				

From above, it can be observed that the preferences marked with asterisks determine cycles in the collective ranking of preferences. With relation to this problem, not long pass investigations carried out by Meyer and Brown (1998) have concluded that whenever dynamic decision-making processes are allowed and carried out in sequence these exhibit some characteristics of chaotic systems.

The stability of the solution is analysed in the way further.

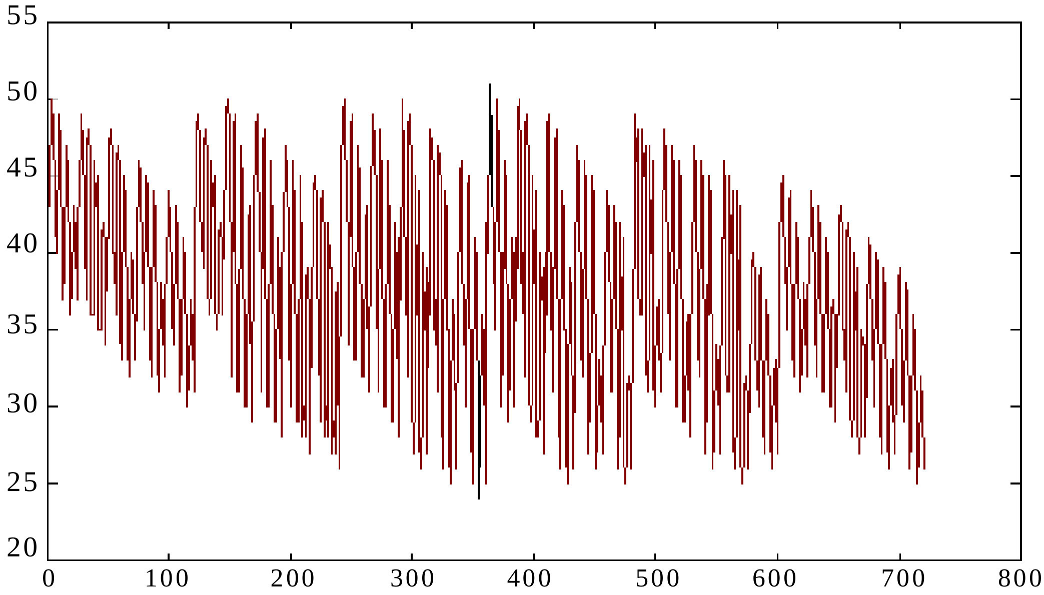

Figure 4. Graphical representation of function of aggregation for U = 24.

Solving numerically Equation (9), for a total energy U = 24, β = 0.4083 is obtained. Moreover, using this value in equations (6), the equilibrium distribution for n = 5, where evidently $n_0 + n_1 \ldots + n_{15} = 5$, is presented in Chart 2.

n_0 = 0.0839	n_4 = 0.8033	n_8 = 0.3234	n_{12} = 0.0181
n_1 = 0.2790	n_5 = 0.7738	n_9 = 0.1915	n_{13} = 0.0058
n_2 = 0.5194	n_6 = 0.6520	n_{10} = 0.1005	n_{14} = 0.0014
n_3 = 0.7152	n_7 = 0.4864	n_{11} = 0.0461	n_{15} = 0.0002

Chart 2. Equilibrium distribution for *U* = 24.

From values in Chart 2, it can be observed that the energetic value distribution pertaining to committee members, as described by the group agreement, fits the demands of statistical equilibrium. That is, $\Theta(\psi_{355}^{6})=(3,9,5,2,5)$ does not comprise values, which under equilibrium conditions assume low probability values, such as 0 or 10,…,15. Figure 11 is a graphical sight of the values in Chart 2. There, it can be noticed that the distribution of energetic values of the committee members behaves in a way that is consistent with equilibrium conditions, even though this example comprises only a reduced umber of agents. Hence, the solution is optimal, and not far of being stable in statistical sense.

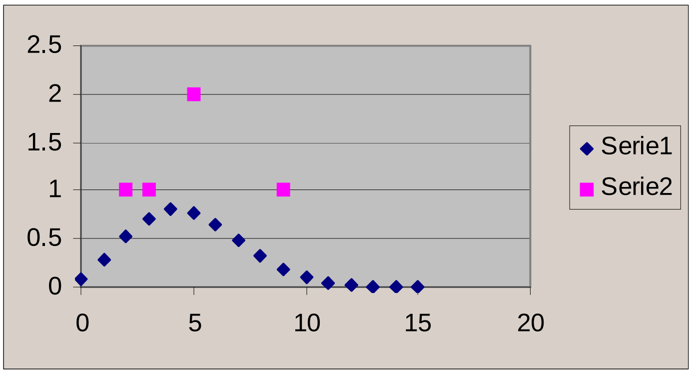

Figure 5. Localization of the energetic value distribution (Serie 2) with respect to the equilibrium distribution for a total energy U = 24 (Serie 1).

NOTE: For the resolution of this case, the solution of the Equation (9) has been found by using the numeric algorithm of secant [Burden and Faires (1990), p.59] by means of a routine implemented in PC-Matlab computer package [Moler **et al** (1984a, 1984b)].

6. FINAL DISCUSSION AND CONCLUDING REMARKS

The concepts that have been used to argue the proposal in this paper allow guaranteeing its rationality at a good extent. Evidently, this approach is restricted by the adequacy of the measurement procedure for preference evaluation and the specific form of the function of aggregation, here represented by Equation (1). Nevertheless, when rational behaviour characterized by a weak order is assumed, then proposed disagreements exist and consequently they can be admitted as subjects of study. It is in those variables which present work concentrates, and starting from the definition of a value or measurement function, some arguments allow to propose an aggregation method whose effectiveness is supported on two nearly axiomatic conditions: optimality and stability, moreover a development firmly sustained in results and concepts kindred to Classical Statistical Mechanics.

The example which is presented at the end of this papers is reasonably useful for noticing that the aggregating method can be used in the same terms than other procedures, like those of simple majority rule; however, the aggregation function that determines the result does not lead to an intransitivity in this case, as it happens when applying simple majority approach. This is due to the very construction of the method makes it impossible to happen. Likewise, statistical equilibrium guarantees the solution to be stable in statistical terms.

Limitations that are early found to the proposal are two-fold. The first one is that evaluation mechanism can eventually drive to a number of solutions (defining a class of indifference) since it is not guaranteed that the solution to which it drives shall be unique. The second one concerns on what should be done when the problem involves many participants and alternatives, because of procedure grows in numeric terms and, therefore, its solution gets complicated in function of the number of operations that must be carried out. In that sense, it seems useful to be plentiful in research for analytic models that avoid the application of numeric procedures; moreover, to improve the structure along with the efficiency of this mechanism is accomplished.

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REFERENCES

ALONSO, M. y E.J. FINN (1980): **Física Vol. III Fundamentos Cuánticos y Estadísticos**, Fondo Educativo Interamericano.

ARMSTRONG, R.L. and J.D. KING (1970): **Mechanics, Waves and Thermal Physics**, Prentice Hall, N.J. USA.

ARROW, K.J. and H. RAYNAUD (1986): **Social Choice and Multicriterion Decision-Making**, MIT Press, USA.

BURDEN, R.L. and J.D. FAIRES (1990): **Análisis Numérico**, Grupo Editorial Iberoamérica, México.

CAMPBELL, D.E. (1976): "Democratic Preference Functions", **Journal of Economic Theory** 12, 259-272.

FISHBURN, P.C. (1971): "A comparative Analysis of Group Decision Methods", **Behavioral Science**, 16, 538-544.

________________ (1973): **The Theory of Social Choice**, Princeton University Press. U.S.A.

FRENCH, S. (1988): "Decision Theory. An introduction to the mathematics of rationality", Ellis Horwood (Wiley). England.

HOGARTH, R. (1991): **Judgement and Choice 2nd The Psychology of Decision**, Wiley USA.

JAYNES, E.T. (1957): "Information Theory and Statistical Mechanics*",* Reprinted from **The Physical Review**, 106(4), 620-630, May 15.

KEENEY, R.L. and RAIFFA, H. (1976): "Decisions with Multiple Objectives: Preferences and Value Tradeoffs", Wiley, USA.

MASARANI, F. and GOKTURK, S. S. (1989): "On the Existence of Fair Matching Algorithms", **Theory and Decision**, 26, 305-322.

MEYER, D.A. and BROWN, T.A. (1998): "Statistical Mechanics of Voting", **Physical Review Letters** 81(8).

MOLER, C.; J. LITTLE; S. BANGERT y S. KLEIMAN (1984a): PC-Matlab Reference Manual V. 1.51 MathWorks, Inc. USA.

____ (1984b): PC-Matlab User's Guide V. 1.51 MathWorks, Inc. USA.

REIF, F. (1969): "Física Estadística", **Berkeley physics course** – volume 5. Reverté, Barcelona.