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A COMPARISON OF APPROXIMATIONS TO PERCENTILES OF THE NONCENTRAL t-DISTRIBUTION

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ABSTRACT
In this paper we review the main proposed approximations to percentiles of the noncentral t-distribution. The approximations are examined for their acuracy over a wide range of values of the parameters of the distribution and for several percentil values. Tables summarizing the approximations are included.

Key words: Cornish-Fishex expansions, Taylor Series, Percentiles

RESUMEN
En este artículo revisamos las principales aproximaciones propuestas para calcular percentiles de la distribución t nocentral. La precisión de las aproximaciones se examina para un amplio rango de valores de los parámetros de la distribución y para varios valores del percentil. Se incluyen tablas que resumen las aproximaciones.

Palabras clave: Expansiones Cornish-Fishex, Series de Taylor, percentiles

1. INTRODUCTION AND SUMMARY

It is widely recognized that the noncentral t-distribution is of considerable theoretical and practical importance in many mathematical and statistical applications. For instance, noncentral t-distribution is useful in evaluating power function of the Student´s t-test (Owen, 1968, 1985; Jonhson **et al.**, 1995, pp. 509-510), calculating confidence interval of the coefficient of variation (Lehmann, 1986, p. 352, Johnson **et al.**, 1995, pp. 510-511; Vangel, 1996), approximating the distribution of sample coefficient of variation and calculation of its percentage points (McKay, 1932; Iglewicz **et al.**, 1968; Vangel, 1996), calculating confidence limits on the proportions in the tail of a normal distribution (Durrant, 1978; Odeh and Owen, 1980), constructing confidence limits on one-sided quantiles and ´tolerance limit´ for the normal distribution (Wolfowitz, 1946; Johnson **et al.**, 1995, pp. 511-512), one-sided tolerance limits for the linear regression (Kabe, 1976), and in the study of acceptance sampling plans involving proportion of defective items (Owe, 1968; 1985). Guenther (1975) describes the use of noncentral t-distribution in testing hypotheses involving the quantiles of two normal populations.

In many applications involving the noncentral t-distribution, one has to compute its percentiles involving the evaluation of the inverse probability functions (see, e.g. Bagui, 1993, 1996). However, the evaluation of such inverse functions is extremely tedious involving slow and expensive techniques of numerical iteration such as the Newton-Raphson procedure (see, e.g., Ralston and Wilf, 1967; Carnahan **et al.**,1969). There are a number of approximations for computing the percentage points of these distributions, at arbitrary probability levels, available in the literature. The applicability of several of these approximations is further enhanced by ease of their computational simplicity. The purpose of this paper is to compare these approximations to determine their accuracy. Some of these approximations were previously investigated by Johnson and Welch (1940), van Eeden (1961), Kramer (1973), Akahira (1995) and Akahira **et al**. (1995). A brief description of each procedure is given and appropriate tables comparing their accuracy, calculated for each procedure, are presented. A more comprenhensive set of tables is given in Sahai and Ojeda (1998).

2. APPROXIMATIONS

The noncentral t-distribution was first derived by Fisher (1931) who also showed how the tables of the standard normal distribution could be used to approximate this distribution. There are several approximations to percentiles of the noncentral t-distribution available in the literature. Some of the important ones are considered here.

In this paper, $t'_{\nu}(\delta)$ will be used to denote a noncentral t-variate with ν degrees of freedom and the noncentrality parameter δ. In addition, $t'_{\nu,\alpha}(\delta)$ will denote its 100α-th percentile defined by:

$$\Pr\left[t'_{\nu}(\delta) \le t'_{\nu,\alpha}(\delta)\right] = \alpha \ .$$

Jennett and Welch (1939), that $X + K\sqrt{X_{\nu}^{2}}$ is approximately normally distributed, where X has a standard normal distribution, X_{ν}^{2} has a chi-square distribution with ν degrees of freedom (X and X_{ν}^{2} are distributed independently) and K is a constant, gave the approximation

$$t'_{\nu,\alpha}(\delta) \cong \frac{\delta b_{\nu} + z_{\alpha}\sqrt{b_{\nu}^{2} + \left(1 - b_{\nu}^{2}\right)\left(\delta^{2} - z_{\alpha}^{2}\right)}}{b_{\nu}^{2} - z_{\alpha}^{2}\left(1 - b_{\nu}^{2}\right)}, \tag{2.1}$$

where

$$b_{\nu} = \frac{\sqrt{2}\ \Gamma\{(\nu+1)/2\}}{\sqrt{\nu}\ \Gamma(\nu-2)}$$

and z_{α} is determined by

$$\Pr\left\{Z < z_{\alpha}\right\} = \int_{-\infty}^{z_{\alpha}} (2\pi)^{-1/2} \exp\left(-\frac{1}{2}z^{2}\right)dz = \alpha \ .$$

Akira (1995) obtained the approximation (2.1) as a special case of the Cornish-Fisher expansion by ignoring terms of higher order than $o(\nu^{-1})$.

Johnson and Welch (1940) simplified the approximation (2.1) leading to the approximation

$$t'_{\nu,\alpha}(\delta) \cong \frac{\delta + z_{\alpha}\sqrt{1 + \frac{1}{2\nu}\left(\delta^{2} - z_{\alpha}^{2}\right)}}{1 - z_{\alpha}^{2}/2\nu} \tag{2.2}$$

Masuyama (1951) obtained values of this approximations using an improved binomial paper. Akahira (1995) obtained the approximation (2.2) as a special case of (2.1) by letting $b_{\nu} \approx 1$ and $1 - b_{\nu}^{2} \approx 1/(2\nu)$.

An approximation intermediate between (2.1) and (2.2) was given by van Eeden (1961) as

$$t'_{\nu,\alpha}(\delta) \cong \frac{\delta b_{\nu} + z_{\alpha}\sqrt{b_{\nu}^{2} + \frac{1}{2\nu}\left(\delta^{2} - z_{\alpha}^{2}\right)}}{b_{\nu}^{2} - \frac{1}{2\nu}Z_{\alpha}^{2}} \ . \tag{2.3}$$

Approximations (2.1), (2.2) and (2.3), however, give real values for $t'_{\nu,\alpha}(\delta)$ only for limited ranges of values of δ and z_{α} (see, e.g., Johnson **et al.**, 1995, p. 521).

For small values of δ and large values of ν(> 20), the simple approximation of the standardized $t'_\nu(\delta)$ variate by a standard normal variate yields the result (Johnson **et al**., 1995, p. 523):

$$t'_{\nu,\alpha}(\delta) \cong \frac{\nu}{\nu-1}\delta b_\nu + z_\alpha \sqrt{\frac{\nu}{\nu-2}\left(1+\delta^2\right) - \frac{\nu^2}{(\nu-1)^2}\delta^2 b_\nu^2} \qquad (2.4)$$

The normal approximation (2.4) is, of course, applicable for very small values of δ and large values of ν and is included here for the sake of completeness.

Cornish and Fisher (1937) (see also Fisher and Cornish, 1960) expansion applied to the distribution of $t'_\nu(\delta)$ (expansion up to and including terms in ν^{-2}) yields the following approximation (van Eeden, 1961; Johnson **et al.,** . 1995, p. 524):

$$\begin{aligned} t'_{\nu,\alpha}(\delta) \cong{}& z_\alpha + \delta + \frac{1}{4\nu}\left[z_\alpha^3 + z_\alpha + \left(2z_\alpha^2+1\right)\delta + z_\alpha\delta^2\right] + \frac{1}{96\nu^2}\left[5z_\alpha^5 + 16z_\alpha^3 + 3z_\alpha\right. \\ &\left. + 3\left(4z_\alpha^4 + 12z_\alpha^2 + 1\right)\delta + 6\left(z_\alpha^3 + 4z_\alpha\right)\delta^2 - 4\left(z_\alpha^2 - 1\right)\delta^3 - 3z_\alpha\delta^4\right]. \end{aligned} \qquad (2.5)$$

Shibata (1981) derived approximation (2.5) from the taylor series expansion of the characteristic function of $t'_\nu(\delta) - \delta$ with a chi-square variate having ν degrees of freedom. Akahira (1995) derived it by applying the Cornish-Fisher expansion and using the characteristic function of a chi-square variate with ν degrees of fredom.

The corresponding Cornish-Fisher expansion applied to the central t-distirbution (δ = 0) gives the result (see, e.g., Sahai and Thompson, 1974):

$$t_{\nu,\alpha} \cong z_\alpha + \frac{z_\alpha^3 z_\alpha}{4\nu} + \frac{5z_\alpha^5 + 16z_\alpha^3 + 3z_\alpha}{96\nu^2} .$$

If these terms in (2.5) are replaced by $t_{\nu,\alpha}$, (2.5) becomes (van Eeden, 1961; Johnson **et al.,** . 1995, p. 524).

$$t'_{\nu,\alpha}(\delta) \cong t_{\nu,\alpha} + \delta + \frac{\delta}{4\nu}\left(1 + 2z_\alpha^2 + \delta z_\alpha\right) + \frac{1}{96\nu^2}\delta\left[3\left(4z_\alpha^4 + 12z_\alpha^2 + 1\right) + 6\left(z_\alpha^3 + 4z_\alpha\right)\delta - 4\left(z_\alpha^2 - 1\right)\delta^2 - 3z_\alpha\delta^3\right] \qquad (2.6)$$

We will call the approximations (2.5) and (2.6) as Cornish and Fisher´s 1st and 2nd approximations respectively:

Azorin (1953), starting from the relationship

$$\operatorname{var}\left(t'_\nu(\delta)\right) = a^2 + b^2\left[E\left(t'_\nu(\delta)\right)\right]^2 \ ,$$

with

$$a = \sqrt{\frac{\nu}{(\nu-2)}}\ , \quad b = \Gamma\left(\frac{1}{2}\nu\right)\sqrt{2\left\{(\nu-2)\left[\left(\tfrac{1}{2}(\nu-1)\right)\right]^2 - 1\right\}^{-1}}$$

and (2.7)

$$E\left(t'_\nu(\delta)\right) = \left(\tfrac{1}{2}\nu\right)^{1/2}\frac{\Gamma\left(\tfrac{1}{2}(\nu-1)\right)\delta}{\Gamma\left(\tfrac{1}{2}\nu\right)}$$

obtained the transformation

$$\frac{1}{b}\sinh^{-1}\left(\tfrac{b}{a}\,t'_\nu(\delta)\right)-\frac{1}{b}\sinh^{-1}\left(\tfrac{b}{a}\,E\left(t'_\nu(\delta)\right)\right), \tag{2.8}$$

which is to be approximated as a standard normal variate. In addition, Azorin (1953) suggested two similar transformations of simpler forms:

$$\sqrt{\nu}\,\sinh^{-1}\left(t'_\nu(\delta)/\sqrt{\nu}\right) \tag{2.9}$$

and

$$\sqrt{\tfrac{2}{3}\nu}\,\sinh^{-1}\left(t'_\nu(\delta)/\sqrt{\tfrac{2}{3}\nu}\right) \tag{2.10}$$

The standarized versions of (2.9) and (2.10), correcting for mean and standard deviation of the transformed variable to terms of order ν^{-1}, are

$$\frac{\sqrt{\nu}\,\sinh^{-1}\left(t'_\nu(\delta)/\sqrt{\nu}\right)-\delta-\tfrac{1}{2}\delta^2\nu^{-1/2}-\tfrac{1}{4}\delta\nu^{-1}}{\sqrt{1+\tfrac{1}{2}\left(2-\delta^2\right)\nu^{-1}}} \tag{2.11}$$

and

$$\frac{\sqrt{\tfrac{2}{3}\nu}\,\sinh^{-1}\left(t'_\nu(\delta)/\sqrt{\tfrac{2}{3}\nu}\right)-\delta-\left(\tfrac{1}{2}\delta^2/\sqrt{\tfrac{2}{3}\nu}\right)}{\sqrt{1+\tfrac{1}{2}\left(1-\delta^2\right)\nu^{-1}}} \tag{2.12}$$

We will call approximations (2.8), (2.11) and (2.12) as Azorin´s 1st, 2nd and 3rd approximations respectively.

There is a considerable degree of skewness in the noncentral t-distribution for large values of δ and small values of ν (Johnson and Welch, 1940). Thus, approximations (2.8), (2.11) and (2.12) are expected to be rather very poor for simultaneously small values of ν and large values of δ. The results of numerical computations show that these approximations are not at all satisfactory; however, they have been included here for the sake completeness. Also, for small values of ν, the quality of approximations deteriorates rather rapidly as δ increases. For very large values of ν, the approximations improve somewhat, bull still are not to be recommended. Only Azorin´s 1st approximation is included in our comparative study.

Laubscher (1960) also considered the transformation (2.8). Furthermore, he proposed two modifications of (2.8) of the form:

$$L_1=\frac{1}{b}\sinh^{-1}\left(\frac{b}{a}\,t'_\nu(\delta)\right)-\frac{1}{b}\sinh^{-1}\left(\frac{b}{a}\,\mu\right)+\frac{1}{2}b^2\mu\,\mu_2^{(-1/2)} \tag{2.13}$$

and

$$L_2=L_1-\frac{1}{6}b^4\mu_2^{(-5/2)}\mu_3\left[2\mu^2-\left(a^2/b^2\right)\right], \tag{2.14}$$

where a, b and $\mu=E\left(t'_\nu(\delta)\right)$ are given as in (2.7) and μ_2 and μ_3 are the second and thirs central moments of $t'_\nu(\delta)$ given by

$$\mu_2 = \frac{\nu\left(1+\delta^2\right)}{(\nu-2)} - \mu^2 \text{ and } \mu_3 = \mu\left[\frac{\nu\left(\delta^2+2\nu-3\right)}{(\nu-2)(\nu-3)} - 2\mu_2\right].$$

The approximations (2.13) and (2.14) are expected to eliminate more bias than (2.8).

In addition, following Laubcher´s (1960) conjecture, we consider the approximations

$$\frac{(2\nu-1)^{1/2}\left[(1/\nu)\,t_\nu'^{\,2}(\delta)\right]^{1/2} - \left[2\left(1+\delta^{-2}\right) - \left(1+2\delta^2\right)/\left(1+\delta^2\right)\right]^{1/2}}{\left[(1/\nu)t_\nu'^{\,2}(\delta) + \left(1+2\delta^2\right)/\left(1+\delta^2\right)\right]^{1/2}} \tag{2.15}$$

and

$$\frac{(1-2/9\,\nu)\left[t_\nu'^{\,2}(\delta)/(1+\delta)^2\right]^{1/3} - \left[1-2\left(1+2\delta^2\right)/9\left(1+\delta^2\right)^2\right]}{\left\{\left[2\,(1+2\delta^2)/9\,(1+\delta^2)^2\right] + (2/9\nu)\left[t_\nu'^{\,2}(\delta)/\,(1+\delta^2)\right]^{2/3}\right\}^{1/2}}\,, \tag{2.16}$$

where each is to be approximated as a standard normal variate.

We will call approximations (2.13), (2.14), (2.15) and (2.16) as Laubscher´s 1st, 2nd, 3rd, and 4th approximations respectively.

Harley (1957) suggested an approximation of $t_\nu'(\delta)$ in terms of a function of the sample correlation coefficient (r), in a random sample of size n = ν + 2 from a bivariate population with correlation coefficient ρ, by the relationship

$$t_\nu'(\delta) = \frac{r}{\sqrt{\left(1-r^2\right)}}\sqrt{\frac{\nu(2\nu+1)}{\left(2\nu+1+\delta^2\right)}} \tag{2.17}$$

where

$$\rho = \delta\sqrt{\frac{2}{\left(2\nu+1+\delta^2\right)}}$$

Approximation (2.17) is of course valid for $\delta \le (2\nu+1)^{1/2}$ and is thus appplicable for only small values of δ. The percentiles of $t_\nu'(\delta)$ can be approximated from the percentiles of r by using relationship (2.17).

One can obtain the percentiles of r by using the Fisher´s Z-transformation

$$Z = \frac{1}{2}\log_e\left(\frac{1+r}{1-r}\right) \tag{2.18}$$

which is considered as approximately normally distributed with mean $\frac{1}{2}\log_e\left(\frac{1+\rho}{1-\rho}\right)$ and variance 1/(n-3). However, following the recommendation of David (1938), we approximate (2.18) by a normal random variable with mean μ and variance σ^2 given by

$$\mu = \frac{1}{2}\log_e\left(\frac{1+\rho}{1-\rho}\right) + \frac{\rho}{2(n-1)}\left\{1+\frac{5+\rho^2}{4(n-1)}\right\}$$

and

$$\sigma^2 = \frac{1}{n-1}\left[1+\frac{4-\rho^2}{2(n-1)}+\frac{22-6\rho^2-3\rho^4}{6(n-1)^2}\right].$$

This approximation is considered to be the most accurate one of all the existing normal approximations of r (see, e.g., Kraemer, 1973).

Another kind of approximation of r was considered by Ruben (1966) who showed that $r/\sqrt{1-r^2}$ is distributed as

$$\left[\,Z+\chi_{n-1}\,\rho/\sqrt{(1-\rho^2)}\,\right]\,/\chi_{n-2}\;, \tag{2.19}$$

where Z is a unit normal variate, χ_ν is a chi-variate with ν degrees of freedom, and Z, χ_{n-1} and χ_{n-2} are mutually independent. For large values of n, χ_{n-1} and χ_{n-2} may be approximated by normal variates using Fisher´s approximation that $\sqrt{2\chi_\nu^2}-2\sqrt{2\nu-1}$ is approximately distributed as a unit normal variate. Using these results, it can be shown that the transformed variate

$$\frac{r\left(1-r^2\right)^{-\frac{1}{2}}\left(n-\frac{3}{2}\right)^{-\frac{1}{2}}-\rho\,\left(1-\rho^2\right)^{-\frac{1}{2}}\left(n-\frac{5}{2}\right)^{\frac{1}{2}}}{\left[1+\frac{1}{2}r^2\left(1-r^2\right)^{-1}+\frac{1}{2}\rho^2\left(1-\rho^2\right)^{-1}\right]^{\frac{1}{2}}} \tag{2.20}$$

has a standard normal distribution.

We will call the approximations of the type (2.17), based on percentiles of r via (2.18), its improved version due to David, and (2.20), as Harley´s, 1^{st}, 2^{nd}, and 3^{rd} approximations respectively.

Merrington and Pearson (1958) gave an approximation to the percentiles of $t'_\nu(\delta)$ based on an approximations by a Pearson Type IV distribution. Let β_1 and β_2 be the moment rations, i.e.,

$$\beta_1 = \frac{\mu_3^2}{\mu_2^3} \text{ and } \beta_2 = \frac{\mu_4}{\mu_2^2}\,,$$

where μ_2, μ_3 and μ_4 denote the second, third and fourth central moments respectively of the noncentral t-distribution. Then, we have

$$t'_{\nu,\alpha}(\delta) \approx \mu + \sigma\,U\left(\beta_1,\beta_2,\alpha\right), \tag{2.21}$$

where $U\left(\beta_1,\beta_2,\alpha\right)$ is the 100α-th percentiles of the standardized Pearson Type IV distribution and μ and σ are the mean and standard deviation respectively of the noncentral t-distribution. The approximation (2.21) is of course applicable for $\nu > 4$ since μ_4 does not exist for $\nu \le 4$. This approximation, however, is not included in our comparative study because of complexity in computing the percentiles of the Pearson Type IV distribution.

Halperin (1963) developed bounds for the percentiles of $t'_{\nu,\alpha}(\delta)$ given by

$$t'_{\nu,\alpha}(\delta) \le \frac{\delta\sqrt{\nu}}{\chi_{\nu,1-\alpha}} + t_{\nu,\alpha} \qquad (\alpha \ge 0.5)$$

and (2.22)

$$t'_{\nu,\alpha}(\delta) \ge \frac{\delta\sqrt{\nu}}{\chi_{\nu,1-\alpha}} + t_{\nu,\alpha} \qquad (\alpha \le 0.43)\ ,$$

where $t_{\nu,\alpha}$ and $\chi^2_{\nu,\alpha}$ denote 100α - th percentiles of central t and χ^2 distributions respectively. Although approximations (2.22) in not of great accuracy, it is included here to investigate the sharpness of the bounds.

Kraemer and Paik (1979) proposed a central t approximation to the noncentral t-distribution by the relationship

$$\Pr\left[t'_\nu(\delta) \le t\right] \approx \Pr\left[\ t_\nu \le t\left(1+\delta^2/\nu\right)^{1/2} - \delta\left(1+t^2/\nu\right)^{1/2}\ \right].$$

Akahira (1995) and Akahira **et al.**, (1995) derived an higher order approximation formula from the Cornish-Fisher expansion for the statistic based on a linear combination of a normal random variable and a chi-square random variable. The approximate percentile $t'_{\nu,\alpha}(\delta)$ derived from the formula is determined by the relationship:

$$\frac{b_\nu t'_{\nu,\alpha}(\delta) - \delta}{\sqrt{1+t^{2\prime}_{\nu,\alpha}(\delta)\left(1-b_\nu^2\right)}} \approx z_\alpha - \frac{t^{3\prime}_{\nu,\alpha}(\delta)\left(z^2_{\alpha-1}\right)}{24\left\{1+t^{2\prime}_{\nu,\alpha}(\delta)\left(1-b_\nu^2\right)\right\}^{3/2}}\left\{\frac{1}{\nu^2}+\frac{1}{4\nu^3}\right\},$$

(2.23)

where b_ν is defined as in (2.1).

Approximation (2.23) gives only an implicit expression and will require an iterative procedure for its solution. Akahira **et al.**, (1995) showed that for a fixed α such that $|z_\alpha| \ge 1$ and for sufficiently large ν which is independent of δ, the solution to (2.23) exists uniquely. The existence of solution is guaranteed when $0.1 \le \alpha \le 0.15$ for $\nu = 1$, $0.03 \le \alpha \le 0.15$ for $\nu = 2$, $0.01 \le \alpha \le 0.15$ for $\nu = 3$, and $0.003 \le \alpha \le 0.15$ for $\nu \ge 4$. Approximation (2.23) is not included in our comparative study. However, Akahira **et al.**, (1995) made a detailed numerical comparison of this along with Jennett-Welch, Johnson-Welch and van Eeden (1st Cornish-Fisher) approximations and found that approximation (2.23) had better numerical precision than others included in the study.

3. RESULTS

The percentiles $t'_{\nu,\alpha}(\delta)$ calculated for various approximations as well as the exact values, for selected values of α, ν and δ, are given in Table 1. On comparing the approximate values of $t'_{\nu,\alpha}(\delta)$ with the exact values one notices some very interesting results. For higher percentiles and small values of ν, Johnson-Welch and normal approximations perform best; however, for smaller values of δ, the Johnson-Welch approximation in less accurate than the normal. As δ increases, the accuracy of Johnson-Welch approximation improves. For moderate to large values of ν, the Jennett-Welch and van Eeden approximations are superior to others; the former being better than the latter. For 50th percentile ($\alpha = 0.5$), the Jennett-Welch and van Eeden approximation are equivalent and perform better than others. In this case, the Johnson-Welch approximation reduces to δ. This fact partially confirms the validity of the computations since they were obtained using the general formulae for all the approximations. For the lower percentiles, the normal approximation performs very poorly. In this case, the van Eeden approximation perfomrs better except when both ν and δ are small and the Jennett-Welch approximation is superior. For all the cases, when ν is sufficiently large, all the approximations compare favorably. Both the 1st and 2nd Cornish-Fisher approximations provide excellent results for moderate to large values of ν and small values of δ. However, both approximations progressively degenerate as λ increases, especially for small values of ν and extreme lower and upper percentiles. In this regard, the performance of the 1st Cornish-Fisher is much worse than the 2nd one. The three approximations due to Azorín perform poorly, especially for small values of ν. Azorín´s 2nd and 3rd approximations show even much poorer performance than the 1st approximation and their results are not included in Table 1.

Similarly, Laubscher´s approximations, especially 3rd and 4th, show a rather poor performance, particularly for higher percentiles. For large values of ν, both Azorín and Laubscher type approximations show a spectacular improvement, but are still not to be recommended. There is not much difference among three approximations of Harley, based on distinct approximations of percentiles of the sample correlation coefficient and the approximations deteriorate rather rapidly for large values of δ. As expected, Halperin´s bounds are not too sharp, and are to be recommended only as a crude approximation. Finally, Kraemer-Pike approximation gives a uniformly poor performance, especially for small values of ν and large values of δ. For lower percentiles, the approximation gets better as ν increases, but is still not to be recommended.

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Table 1. Approximate and exact percentiles of the noncentral t-distribution

$\alpha = 0.05$

ν	APPROXIMATIONS	δ					
		1	4 -	7	12	25	32
4	Jennet-Welch	-0.7411	2.1020	4.3267	7.7942	16.5568	19.9033
	Johnson-Welch	-0.6936	1.9891	4.1023	7.3954	15.7145	18.8914
	van Eeden	-0.7458	2.0824	4.2754	7.6944	16.3380	19.6396
	Normal Approximation	-1.3103	0.1152	0.8794	1.9010	4.2925	5.1877
	1st Cornish-Fisher	-0.7384	2.1824	7.5897	54.0981	1130.31	2394.57
	2nd Cornish-Fisher	-0.7384	1.9281	3.3582	4.4074	15.1171	28.9562
	1st Azorín	-1.1456	1.4591	3.3179	6.1314	13.1484	15.8195
	1st Laubscher	-1.3163	1.0992	2.7857	5.2813	11.4376	13.7729
	2nd Laubscher	-1.3379	1.1301	2.8193	5.3169	11.4713	13.8066
	3rd Laubscher	0.0974	2.1492	4.2658	7.6311	16.1752	19.4412
	4th Laubscher	-0.3424	2.1079	4.2929	7.7189	14.1008	17.3473
	1st Harley	-0.9217	---------	---------	---------	---------	---------
	2nd Harley	-0.7155	---------	---------	---------	---------	---------
	3rd Harley	-0.7816	---------	---------	---------	---------	---------
	Halperin	-1.4825	0.4654	2.4133	5.6598	14.1008	17.3473
	Kraemer-Paik	0.9219	1.0793	2.4710	4.5713	9.8061	11.7985
	Exact	-0.7389	2.0801	4.2453	7.6163	16.1484	19.4090

'---------' designates undefined values.