A MODIFIED PENALTY EMBEDDING FOR LINEAR COMPLEMENTARITY PROBLEMS[1]

Sira M. Allende Alonso, Facultad de Matemática y Computación, Universidad de La Habana
Jürgen Guddat and Dieter Nowack, Institut für Mathematik, Humboldt Universität zu Berlin

ABSTRACT
We propose a modified penalty embedding for solving complementarity problem (LCP). This embedding is a special one parametric optimization problem P(t), t∈[0,1]. Under the condition (A3) (a modified Enlarged Mangasarian Fromovitz Constrait Qualification), (A4) (P(t) is Jongen- Jonker -Twilt regular) and two technical assumptions (A1) and (A2) there exists a path in the set of stationary points connecting the chosen starting point for P(0) with a certain point for P(1) and this point is a solution for (LCP). The path may include types of singularities, namely points of Type 2, Type 3 and Type 4 in the class of Jongen-Jonker-Twilt. We can follow this path by using pathfollowing procedures (program package PAFO) only. We do not have any assumption with respect to the matrix B in the description of the (LCP). The assumption (A4) will justified by two theorems. An illustrative example shows that points of Type 2 and 3 could appear.

Key words: Linear complementarity problem, penalty embedding, non degenerate critical points, singularities, Jonge-Jonker-Twilt regularity, Mangasarian Fromowitz Constraint, path-following methods.

RESUMEN
Se propone un embedding de penalidad modificado para resolver el problema de complementariedad lineal (PCL). Esta inmersión es un problema de optimización paramétrica especial P(t), t ∈ [0,1]. Bajo la condición (A3) (a modified Enlarged Mangasarian Fromovitz Constraint Qualification), (A4) (P(t) es Jongen- Jonker -Twilt regular) y las hipótesis (A1) y (A2) existe un camino sobre el conjunto de puntos estacionarios conectando el punto inicial seleccionado para P(0) con un tal punto para P(1) y este punto es una solución de (LCP). El camino puede incluir singularidades, denominadas de Tipo 2, Tipo 3, Tipo 4 por-Jonker-Twilt. No se establece condición sobre la matriz B en la descripción del problema. El programa PAFO posibilita seguir el camino descrito. La hipótesis (A4) es justificada por dos teoremas. Un ejemplo ilustra el procedimiento.

Palabras clave: Problema de complementariedad Lineal, inmersión de penalidad, puntos críticos no degenerados, singularidades, Jonge-Jonker-Twilt-regularidad, Condición de regularidad de Mangasarian Fromowitz, método de continuación.

1. INTRODUCTION

Let B be an $n \times n$ -matrix, $q \in \Re^n$, and

$$M^L := \{x \in \Re^n \mid Bx + q \geq 0,\ x^T Bx + q^T x = 0,\ x \geq 0\}.$$

We consider the well known linear complementarity problem (we refer e.g. to Burke-Xu (1998), Cottle, R.W. **et al**. (1992), Ferris, M.C. **et al**. (1997), Fischer, A. (1995), Kojima, M. **et al**. (1991), Stoer, J. **et al**. (1998) and the papers cited there)

(LCP) Find a point $\hat{x} \in M^L$ (1.1)

There are many interesting applications of this problem (cf. for instance Ferris M.C. **et al**. (1997)). If we introduce

$$B = \begin{pmatrix} b^{1T} \\ \vdots \\ b^{nT} \end{pmatrix} \quad \text{with} \quad b^j \neq 0,\ j = 1,\dots n, \text{ and } b^j \in \Re^n,$$

[1]This research was supported by the Deutsche Forschungsgemeinschaft under grant Gu 304/12-1 and by the Deutscher Akademischer Austausch Dienst under grant: 414-A/98/17000

hen we can write M in the following form:

$$M^L := \{x \in \Re^n \mid b^{jT} x + q_j \geq 0,\ x^T Bx + q^T x = 0,\ x_j \geq 0,\ j \in J\},$$

where J: = {1,..., n}

We assume that

(A1) $M^L \neq \emptyset$

Let $E(x_1, p) := \{x \in \Re^n \mid \|x - x_1\|^2 \leq p\}$, where $x_1 \in \Re^n$ is an arbitrarily chose and fixed vector and $p \in \Re$ with $p > 0$.

Then there exists a $p_0 > 0$ such that $M^L \cap E(x_1, p) \neq \emptyset$ for all $p > p_0$ (1.2)

If M^L is compact, then we even have: $M^L \subseteq E(x, p)$ for all $p > p_0$.

Instead of the (LCP) (cf. (1.1) we now consider the following optimization problem

$$(P^L) \quad \min\left\{\frac{1}{2}(x - x_0)^T A(x - x_0) \,\middle|\, x \in M^L\right\} \tag{1.3}$$

where A is a symmetric n×n matrix ($A \in \Re^{n(n+1)-/2}$), here the space of symmetric n×n matrices is identical with $\Re^{n(n+1)-2}$ and $x_0 \in \Re^n$. We follow the concept of modified penalty embeddings described in Gollmer R. **et al**. (1993), Gómez, W. **et al**. (2000), Guddat J. **et al**. (1997), Guddat J. **et al**. (1990)) (first used in Gfrerer, H. **et al**. (1985)). The problem (P^L) will be embedded by

$$P^L(t) \qquad \min\left\{\frac{1}{2}(x - x_0)^T A(x - x_0) + \|w - w_0\|^2 + \|v - v_0\|^2 \,\middle|\, (x, w, v) \in M^L(t), t \in [0,1]\right\} \tag{1.4}$$

where

$$M^L(t) = \left\{(x, w, z, v) \in \Re^n \times \Re^n \times \Re^n \times \Re \,\middle|\, \begin{array}{c} t(b_j^T x + q_j) + (1-t)(w_j - (w_1)_j) \geq 0, j \in J \\ tx_j + (1-t)(z_j - (z_1)_j) \geq 0, j \in J \\ t(x^T Bx + q^T x) + (1-t)(v_j - v_1) = 0 \\ p - \|x - x_0\|^2 - \|w - w_2\|^2 - \|z - z_2\|^2 - (v - v_2)^2 \geq 0 \end{array}\right\} \tag{1.5}$$

where $J := \{1,...,n\}, x_0, x_1 w_0, w_1, w_2 z_0, z_1, z_2 \in \Re^n, v_0 \in \Re$ are fixed, p > 0, s > 0 sufficiently large.

We use the pathfollowing procedure for a suitable chosen w_0, w_1, z_0, z_1 obtaining a very good starting situation for t = 0. If we achieve t = 1, we have a solution of the (LCP). The use of pathfollowing methods for (LCP) (cf. e.g. Burke-Xu (1998), Cottle, R.W. **et al**. (1992), Kojima M., **et al**. (1991), Stoer, J. **et al**. (1998) and the papers cited there) is not new. Modified penalty embeddings (cf. above) are not new either. What is new is the application of this embedding to the (LCP). We will see that we achieve t = 1 using path following procedure only, without any assumption concerning the matrix B, like in Burke-Xu (1998), Fischer, A. (1995), Kojima, M. **et al**. (1991), Stoer, J. **et al**. (1998) The matrix B could also be indefinite. This is the real advantage of this approach. From this point of view it is not necessary to compare our pathfollowing procedure with others for (LCP). Section 2 includes a summary of the theoretical background and a short description of the program package PAFO. In Section 3 important properties of P(t) (i.e., the starting situation and the singularities that may appear) will be discussed. In Section 4 theorems justifying the chosen approach are presented. An illustrating example is given in Section 5, and it shows us that point so Type 2 and 3 could appear , we achieve t = 1 and the matrix is indefinite. We were also successful with all the other examples calculated.

2. THEORETICAL BACKGROUND AND THE PROGRAM PACKAGE PAFO

First , we present a very short version of 2.5, 2.6 in Guddat J. **et al**. (1990). We consider the general one-parametric problem:

$$P(T) \qquad \min\{f(x,t) \mid x \in M(t)\},\ t \in \Re \text{ resp. } t \in [0,1], \qquad (2.1)$$

where
$$M(t) = \{x \in \Re^n \mid h_i(x,t) = 0, i \in I, g_j(x,t) \geq 0, j \in J\}$$

and

$$f, h_i, g_j \in C^3(\Re^n \times \Re \times \Re),\ i \in I,\ j \in J.$$

Furthermore , we introduce the following notations

$$\sum\nolimits_{gc} := \{(x,t) \in \Re^n \times \Re \mid x \quad \text{is a generalized critical point}^1 \text{ of } P(t)\},$$

$$\sum\nolimits_{stat} := \{(x,t) \in \Re^n \times \Re \mid x \quad \text{is a stationary point } P(t)\},$$

$$\sum\nolimits_{loc} := \{(x,t) \in \Re^n \times \Re \mid x \quad \text{is a local minimizer of } P(t)\},$$

$$H := (h_1,...,h_m)^T, G := (g_1,...,g_s)$$

The Linear Independence Constraint Qualification (briefly LICQ) is satisfied at $x \in M(\bar{t})$ if the vectors $D_x h_i(\bar{x},\bar{t}), i \in I, D_x g_{ji}(\bar{x},\bar{t}),\ j \in J_0(\bar{x},\bar{t})$ are linearly independent, where $J_0(x,t) := \{j \in J \mid g_j(x,t) = 0\}$.

The Mangasarian-Fromovitz Constraint Qualification (briefly MFCQ) is satisfied at $x \in M(\bar{t})$ if:

(MF1) $D_x h_i(\bar{x},\bar{y}),\ i \in I$ are lineary independent,

(MF2) There exists a vector $\xi \in \Re^n$ with

$$D_x h_i(\bar{x},\bar{y})\xi = 0,\ i \in I^2,$$

$$D_x g_j(\bar{x},\bar{y})\xi > 0,\ j \in J_0(\bar{x},\bar{y}).$$

Next, we cite our short characterization from Gómez, W. **et al**. (2000), Guddat, J. **et al**. (1990) of the class **F** introduced by Jongen, Jonker and Twilt (Jongen **et al**.(1986)). In Jongen, H.Th. **et al**. (1986) the local structure of Σ_{gc} is completely described if (f,H,G) belongs to a C^3_s-open and dense subset **F** of $C^3_s(\Re^n \times \Re, \Re)^{1*ms}$, where C^3_s denotes the strong (or Whitney-) C^3-topology (see Guddat J., **et al**. (1990), too).

If (f,H,G) $\in$ **F**, then Σ_{gc} can be divided into 5 types.

Type 1: A point $\bar{z} = (\bar{x},\bar{t}) \in \Sigma_{gc}$ is of Type 1 (non-degenerate critical point) if the following conditions are satisfied:

There exists $\bar{\lambda}_i, \bar{\mu}_j \in \Re, i \in I,\ j \in J_0(\bar{z})$ with

$$\left.\left(D_x f + \sum_{i \in I} \bar{\lambda}_i D_x h_i + \sum_{j \in J_0(\bar{z})} \bar{\mu}_{ji} D_x g_j\right)\right|_{z=\bar{z}} = 0 \qquad (2.2)$$

LICQ is satisfied at $\bar{x} \in M(\bar{t})$ (2.3a)

(therefore $\bar{\lambda}, \bar{\mu}_{ji}$, $i \in I$, $j \in J_0(\bar{z})$ are uniquely defined)

$$\bar{\mu}_{ji} \neq 0,\ j \in J_0(\bar{z}) \tag{2.3b}$$

$D_x^2 L(\bar{x}.\bar{t}) \Big|_{T(\bar{z})}$ is non singular (2.3c)

where $D_x^2 L$ is the Hessian of the Lagrangian

$$L(x,t) = f(x,t) + \sum_{i \in I} \bar{\lambda}_i h_i(x,t) + \sum_{j \in J_0(\bar{z})} \bar{\mu}_{ji} g_j(x,t)$$

and the uniquely determined numbers $\bar{\lambda}_i, \bar{\mu}_j$ are taken from (2.2). Furthermore, $T(z) = \{\xi \in \Re^n \,|\, D_x h_i\,(z)\xi = 0,$ $i \in I$, $D_x g_j(z)\xi = 0$, $j \in J_0(z)\}$ is the tangent space at z and $D_x^2 L(z)\,|_{T(z)}$ represents $V^T D V_x^2 L V$, where V is a matrix whose columns form a basis of T(z).

The set Σ_{gc} is the closure of the set of all points of Type 1, the points of the Types 2--5 constitute discrete subset of Σ_{gc}. The points of the Types 2--5 represent three basic degeneracies:

Type 2 -- violation of (2.3b)

Type 3 -- violation of (2.3c)

Type 4 -- violation of (2.3a) and $|I| + |J_0(x,t)| - 1 < n$

Type 5 -- violation of (2.3a) and \$$|I| + |J_0(x,t)| = n + 1$.

Remark 2.1 In Section 4 we need a complete description of a point of Type 4 (cf. Gómez, W. **et al**. (2000))

Let α^* be fixed $J_0(x,t) = \{1,\ldots\ldots, p\}$

$D_x g_p(x^*,t^*) \in$ span $\{D_x h_i(x^*,t^*)$, $i \in I$, $D_x g_j(x^*,t^*)$, $j = 1,\ldots\ldots\ldots,p - 1$

$(x^*,t^*) \in \sum_{gc}^4$, if the following conditions are satisfied:

a) $1 \leq m + p \leq n$ and it holds $(D_x h\_1(x^*,t^*),\ldots,D_x h_m(x^*,t^*), D_x g\_1(x^*,t^*),\ldots.,D_x g_p(x^*,t^*)) = m + p - 1$

b) $\alpha^*_{m+j} \neq 0$ for all $j \in \{1,\ldots\ldots\ldots, p\}$ where α^* is fixed and defined in

$$\sum_{i=1}^{m} \alpha_i^* D_x h_i(x^*,t^*) + \sum_{j=1}^{p} \alpha^*_{j+m} D_x g_i(x^*,t^*) = 0$$

$$\alpha^*_{j+m} \neq 0, \qquad j = 1 \ldots p$$

c) $(t^*,\alpha_1^*,\ldots,\alpha_{m+p-1}^*, x^*,0) \in R^{n+m+p+1}$ is a non-degenerate critical point of the problem.

min $t|(x,\alpha,t,\alpha_0)$: $\boldsymbol{G}(x,\alpha,t,\alpha_0) = 0\}$ where

$$
\boldsymbol{G}(x,\alpha,t,\alpha_0)=\begin{pmatrix} D_x\left(\alpha_0 f(x,t)-\sum_{i\in I}\alpha_i h_i(x,t)-\sum_{j=1}^{p-1}\alpha_{m+j}g_j(x,t)-\alpha_{m+p}g_p(x,t)\right)\\ h_1(x,t)\\ \vdots\\ h_m(x,t)\\ g_1(x,t)\\ \vdots\\ g_p(x,t)\end{pmatrix}
$$

The following theorem provides a special perturbation of (f,H,G) with additional parameters that can be chosen arbitrarily small such that the perturbed function vector belongs to the class F. Let the space of symmetric $n\times n$-matrices be identified by $R^{\frac{n(n+1)}{2}}$.

Let $\sum_{gc}^{\nu}$ $\nu \in \{1,...,5\}$ be the set of g.c.- points of Type ν. The class *F* is defined by

$$
F = \{(f,H,G) \in C^3(R^n \times R, R^{1+m+s})\ \Sigma_{gc} \subset \cup_{\nu=1}^{5} \sum\nolimits_{gc}^{\nu}
$$

Theorem 2.2: Let (f, H, G) $\in C^3(R^n\times R,R^{1+m+s})$. Then, for almost all (b,A,c,D,e,F) $\in R^n\times R^{\frac{n(n+1)}{2}}\times R^m\times R^{nm}\times R^s\times R^{ns}$, $(f(x,t) + b^T x + x^T Ax, H(x,t) + c + Dx, G(x,t) + e + Fx) \in F$.

Here "almost all" means:

each measurable subset of $\{(b,A,c,D,e,F)\backslash(f(x,t) + b^T x + x^T Ax, H(x,t) + c + Dx, G(x,t) + e + Fx) \notin F\}$ has the Lebesgue-measure zero.

Definition 2.3: Let $K \subseteq R \cup \{\infty\}$. The problem P(t) is called regular in the sense of Jongen-Jonker-Twilt, briefly JJT-regular, (with respect to K) if:

$$
(f,H,G) \in F|_K\ ((\backslash R^n\times K) \cap \Sigma_{gc} \subset \cup_{\nu=1}^{5} \Sigma^{\nu}{}_{gc})
$$

The following theorem is essential for our analysis.

Theorem 2.4 (follows from Gefrerer,H. **et al**. (1985)). *We assume*

(C1) M(t) is non-empty and there exists a compact set C with $M(t) \subseteq C$ for all $t \in [0,1]$.

(C2) P(t) is JJT-regular with respect to [0,1].

(C3) There exists a $t_1 > 0$ and a continuous function $x:[0,t_1) \to R^n$ such that x(t) is the unique stationary point for P(t) for $t \in [0, t_1)$.

(C4) MFCQ is satisfied for all $x \in M(t)$ for all $t \in [0,1]$.

Then there exists a PC^2-path in Σ_{stat} that connects $(x^0,0)$ with some point $(x^1,1)$.

Now we describe the modified penalty embedding used in Gómez W. (1997), Gómez W. **et al**. (2000), Guddat, J. **et al**. (1997) and denoted by $P^p(t)$ for the general optimization problem

$$
(P)\ \min \{f(x)\ \ |x \in M\}, \tag{2.4}
$$

where

$$
M := \{x \in R^n\ \ |\ h_i(x) = 0,\ i \in I,\ g_j(x) \geq 0,\ j \in J\}
$$

and h_i, $g_j \in C^3(R^n, R)$, $i \in I$, $j \in J$

$$P^p(t): \min\{tf(x) + (1-t)(x-x^0)^T A(x-x^0) + (v-v^0)^T C(v-v^0) + (w-w^0)^T D(w-w^0) \mid (x,v,w) \in M^p(t)\}, \quad t \in [0,1] \tag{2.5}$$

where

$$M^P(t) = \left\{ (x,w,v) \in \Re^n \times \Re^m \times \Re^s \;\middle|\; \begin{array}{c} th_i(x) + (1-t)(v_i - v_i^0) = 0, i \in I \\ tg_j + (1-t)\ldots(w_j - (w_{j_1}^1) \geq 0, j \in J \\ p - \|x - x_0\|^2 \geq 0 \\ q - \|w - w_2\|^2 - \|z - z_2\|^2 - (v - v_2)^2 \geq 0 \end{array} \right\} p,q > 0$$

sufficiently large.

We observe that $P^L(t)$ is a special parametrization of $P^p(t)$ (cf. (2.5)) and $P^p(t)$ is a special parametrization of P(t) (cf. (2.1)). The main properties of this embedding are included in Theorem 1.1 in Gollmer R. **et al**. (1993), (good starting situation, $M^p(t) \neq \emptyset$ for all $t \in [0,1]$, equivalence of $P^p(1)$ and (P).

To apply Theorem 2.3 we ask for a sufficient condition on the feasible set M of the original problem (P) (cf. (2.4)) that (C4) in Theorem 2.3 is satisfied. Here we follow Gómez, W. **et al**. (2000) and Guddat, J., **et al**. (1997). Under the assumption that M is non-empty we fix $x^0 \in R^n$ and p in such a way that $M \cap E(x^0,p) \neq \emptyset$ where $E(x^0,p):= \{x \in R^n n \mid \backslash |x-x^0\backslash|^2 \leq p\}$.

Definition 2.5: The Enlarged Mangasarian-Fromovitz Constraint Qualification (EnMFCQ) is satisfied in M if, for all $x \in E(x^0,p)$

EnMFCQ1 $D_x h_i(x)$, $i \in I$, are linearly independent

EnMFCQ2 There exists a vector $\$\xi \in R^n$ with the following properties

$$h_i(x) + D_x h_i(x)\,\xi = 0, \; i \in I$$

$$g_j(x) + D_x g_j(x)\xi > 0, \forall j \in J \text{ with } g_j(x) \geq 0,$$

$$-2(x-x^0)^T\xi > 0, \text{ if } \; ||x-x^0||^2 = p.$$

In Gómez, W. **et al**. (2000) the following Mangasarian-Fromovitz vectors (MF-vectors) are used:

For all $\bar{x} \in M^p(0)$

$$\eta := (-(\bar{x} - x^0), 0, \frac{1}{2}(w_1+w_0) - \bar{w}) \tag{2.6}$$

For all $(\bar{x},\bar{v},\bar{w}) \in M^p(t)$, $t \in (0,1)$:

Let ξ be a vector that realizes the EnMFCQ2. We fix a number

$\gamma g_j(\bar{x}) + Dg_j(\bar{x})\xi < 0$ for all $j \in J^{pos}$ where $J^{pos} := \{j \in J \mid g_j(\bar{x}) > 0\}$. With this number we define the following $w^\eta \in \Re^s$, where the j-th component has the following value:

$$w_j^{\eta} := \begin{cases} -\left(\bar{w}_j - (w_1)_j\right) & \text{if} \quad j \in J \setminus \left(J^{pos} \cap \hat{J}_0(\bar{y},\bar{t})\right) \\ -\gamma\left(\bar{w}_j - (w_1)_j\right) & \text{if} \quad j \in \left(J^{pos} \cap \hat{J}_0(\bar{y},\bar{t})\right) \end{cases} \tag{2.7}$$

where $\hat{J}_0(\bar{y},\bar{t}) = \left\{ j \mid \bar{g}_j(y,t) = 0 \right\}$ and

$$\eta ::= \left(\eta, -(\bar{v} - v_0), w_\eta\right)$$

is a MF-vector.

On the program package PAFO (this is a very short version of Chapter 4.5 and 5.2 in Guddat, J. **et al**. (1990)).

PAFO (cf. Gómez, W. **et al**. (2000), cf. Guddat, J. **et al**. (1990)) is based on a pathfollowing method (called PATH III in Guddat, J. **et al**. (1990) Chapter 4.5) and jumps (called JUMP I in Chapter 5.2 Guddat, J. **et al**. (1990) and JUMP II in Chapter 5.3 Guddat, J. **et al**. (1990))

We explain the main ideas of PATH III, but not those of JUMP I, II as we do not need them here.

PATH III

This algorithm computes a numerical description of a compact connected component in Σ_{gc}, i.e., in particular it finds a discretization of an interval $[t_A,t_B]$, $t_A < 0 < t_B$ (not necessarily $[t_A, t_B] \supset [0,1]$), and corresponding g.c. points starting at $(x_0,0) \in \Sigma_{gc}$(cf. (A2)). The algorithm is based on the active index set strategy and is a so-called predictor-corrector scheme if the active index set is constant. A Newton corrector is used.

The main point of the approach consists in the computation of the new index sets for the possible continuations at points of Type 2 and 5. In our application only points of Type 2 could be appear. This is done easily without any numerical problem.

We note that we do not have any numerical difficulties walking around turning points of the Types 3 or 4. In our application only points of Type 3 could appear.

Remark 2.6

If there exists a PC^2-path connecting $(x_0,0)$ and a point $(x^*,1)$, PAFO constructs a finite number of predictor steps in [0,1], i.e., a discretization

$$0 = t_0 \leq \ldots \leq t_i \leq t_{i+1} \leq \ldots \leq t_N = 1$$

and, by corrector steps using Newton-like methods, corresponding approximations $\bar{x}(t_i)$ of stationary points

$$x(t_i),\ i = 1,\ldots,N,$$

where the rate of convergence will be at least superlinear and the points $\bar{x}(t_j)$ will be obtained by a finite number of Newton-like steps.

3. PROPERTIES OF THE MODIFIED PENALTY EMBEDDING

We consider the problem (P^L) (cf. (1.3)) and the modified penalty embedding $P^L(t)$, $t \in [0,1]$ (cf. (1.4)). We follow here Section 8.2.2 in Gollmer, R. **et al**., 2001) for the problem $P^p(t)$ and we consider the special problem $P^L(t)$. We choose

(A2) x_0, x_1, w_0, $w_1 \in \Re^n$ with $w_0 > w_1$, z_0, z_1, $z_2 \in \Re^n$ with $z_0 > z_1$, and:

$$||x_0 - x_1||^2 + ||w_0 - w_2||^2 + ||z_0 - z_2||^2 + ||v_0 - v_2||^2 < p$$

Theorem 3.1. Let (A1) and (A2) be satisfied. Then we have the following properties for $P^L(t)$

(i) If we choose the matrix A to be positive definite, then (x_0,w_0,z_0,v_0) is a global minimizer, the unique stationary point for \$$P^L(0)$\$. Furthermore, (x_0,w_0,z_0,v_0) is a nondegenerate critical point for $P^L(0)$.

(ii) $M^L(t)$ is non-empty for all $t \in [0,1]$.

(iii) Let Π_x be the orthogonal projection of $\Re^n \times \Re^n \times \Re^n$ onto $\Re^n$. Then $\Pi_x(M^L(1)) = M^L \cap E(x_0,p)$.

Now we consider a modification of the EnMFCQ. (Definition 2.5).

Definition 3.2 The Modified EnMFCQ is satisfied if it holds for all x Î $E(x_0, p) \backslash M^L$ that:

EnMFCQ 1 $(B+B^T)x + q \neq 0$

EnMFCQ 2 There exists a vector $\eta \in \Re^n$ with the following properties

a) $x^TBx + q^Tx + ((B+B^T)x + q)^T \eta = 0$

b) $b^{jT}x + q_j + b^{jT}\eta > 0$, $j \in J$ with $b^{jT}x + q_j \leq 0$

c) $x_j + e^{jT} \eta > 0$, $j \in J$ with $x_j \leq 0$ (e^j is the j-th unit vector)

d) $-2(x-x_0)^T\eta > 0$ if $||x-x_0||^2 = p$

Remark 3.3 By a geometrical interpretation of the EnMFCQ it will be obvious that a violation of theEnMFCQ is possible, but in exceptional cases only. (If a point belonging to M^L is obtained as feasible solution of M(t), for some $t \in [0,1]$, then the problem has been solved).

Theorem 3.4 We assume:

(A3) EnMFCQ is satisfied.

Then the MFCQ is satisfied for all $y \in M^L(t)$ for all $t \in [0,1)$, where y = (x,w,z,v)

Proof: We use the same MF-vectors η as used in ((2.6) and (2.7)) specialized for $M^L(t)$ for t = 0 and $t \in (0,1)$. □

Using the Theorems 2.4, 3.1 and 3.4 we obtain the following summarizing result:

Corollary 3.5 If we choose the matrix A to be positive definite, and if (A1), (A2), (A3), and

(A4) $P^L(t)$ is JJT-regular with respect to [0,1] are satisfied, then there exists a PC^2-path in Σ_{stat} that connects $(y^0,0)$ with some point $(y^*,1)$.

The following remark concludes our investigation.

Remark 3.6

Using the properties of the functions defining $M^L L(t)$ we can apply Theorem 3.1.1 from Bank,B. **et al**. (1982). Then we obtain that the point-to-set mapping t $t \to M^L(t)$ is closed in a neighbourhood U_ε (1), i.e., for each sequence $\{t_k\}$ with $t_k \to 1$ and each sequence $\{y^k\}$ with $y^k \in M(t_k)$ there exists a convergent subsequence of $\{y^k\}$ and its limit y' belonging to $M^L(1)$, i.e, $y' \in M^L$. This is the reason why we are successful with the pathfollowing procedure.

4 JUSTIFICATION THEOREMS

Let $[[C^3_s]^{3+2n} = \{f:B \to \Re^{3+2n} \mid B = \{y:||y||^2 \le p\}, f \in C^3\}$ with the strong topology. We consider the embedding

$P_\chi(t)$: $\min\{(x-x_0)^T A(x-x_0) + (v-v_0)^2 + ||w-w_0||^2 + ||z-z_0||^2$

s.t.

$$
\begin{aligned}
& t(x^T Bx+q^T x)+(1-t)(v-v_1) = 0, && (1)\\
& tx + (1 - t)(z - z_1) \ge 0, && (2)\\
& t(Bx + q) + (1 - t)(w - w_1) \ge 0 , && (3)\\
& ||x - x_1||^2 + (v - v_2)^2 + ||w - w_2||^2 + ||z - z_2||^2 \le p && (4)
\end{aligned}
\qquad (4.1)
$$

Theorem 4.1 For almost all $\chi = (x_0;x_1;v_0;v_1;v_2;w_0;w_1;w_2;z_0;z_1;z_2;A) \in \Re^{11n+[n(n+1)]/2}$, $P_\chi(t)$ is JJT-regular.

Proof: Let $(x,v,w,z,t) = y \in \sum_{gc}$ where the LICQ is satisfied. We will suppose that the compactification constraint is active. In the other case the proof is analogous. That means:

$$
\begin{aligned}
& 2A(x - x_0) + \lambda t[(B + B^T)x + q] - tB_1\mu^1 - tI_2\mu^2 - 2\mu(x - x_1) = 0\\
& 2(w - w_0) -(1 - t)I_1\mu^1 - 2\mu(w - w_2) = 0\\
& 2(z - z_0 - (1 - t)I_1\mu^2 - 2u(z - z_2) = 0\\
& 2(v - v_0 - (1 - t)\lambda - 2\mu(v - v_2) = 0\\
& t(x^T Bx + q^T x) + (1 - t)(v - v_1) = 0\\
& tx_i + (1 - t)(z_i - (z_1)_i 1);\ i \in J_0(y,t) = 0\\
& t(B_1 x + q) + (1 - t)(w - w_1) = 0\\
& |x - x_1||^2 + (v - v_2)^2 + ||w - w_2||^2 + ||z - z_2||^2 = p
\end{aligned}
\qquad (4.2)
$$

where B_1, I_1 are the sub-matrices of B,I_n, corresponding to the active constraints of inequalities (2). I_2 is analogously defined in the case of inequalities (3). As we want to study the characteristics of the critical points: the number of multipliers that are zero, and the rank of the Hessian of the Lagrangian, we will consider the following system:

$$
\begin{aligned}
& D_{x,w,z,v,\lambda,\mu^1,\mu^2,\mu,} L\left(x, w, z, v, \lambda, \mu^1, \mu^2, \mu, t, \chi\right) = 0 && (1)\\
& D^2_{x,w,z,v,\lambda,\mu^1,\mu^2} L\left(x, w, z, v, \lambda, \mu^1, \mu^2, \mu, t, \chi\right) = \gamma && (2)\\
& \gamma_1 - \gamma_2^T (\gamma_3)^{-1} \gamma_2 = 0 && (3)\\
& \mu^1_j = 0, && j \in J_0(y,t) \setminus J_+(\mu^1)\\
& \mu^2_j = 0, && j \in J_0(y,t) \setminus J_+(\mu^2)\\
& \mu = 0
\end{aligned}
\qquad (4.3)
$$

where $J_+(\mu^i) = \{j \in J\ \ \mu^i_j > 0,\ i = 1,2\}$

The system (1) presents the equations corresponding to the characterization of the generalized critical point, in (2) the equations are taken using the symmetry of the Hessian. If $\gamma_1 \in \Re^k$, (2) and (3) imply that the Hessian has k eigenvalues that are zero. The last three systems represent the null multipliers corresponding to the active constraints. We are going to apply the Sard parametric Theorem. The Jacobian U of the system (4.2), depending on some of the parameters χ variables and multipliers, is defined by:

$$U = (D_{\text{variables multipliers}},\ D_{\text{parameters}})$$

$$D_{\substack{\text{variables}\\\text{multipliers}}} = \begin{array}{ccccccccc}
D_x & D_w & D_z & D_v & D_\lambda & D_{\mu 1} & D_{\mu 2} & D_\mu & D_\gamma \\
\otimes & 0 & 0 & 0 & \otimes & \otimes & \otimes & -\bar{x}_x & 0 \\
0 & \bar{\mu} I_n & 0 & 0 & 0 & \begin{matrix} tI_1 \\ 0 \end{matrix} & 0 & -\bar{w} & 0 \\
0 & 0 & \bar{\mu} I_n & 0 & 0 & 0 & \begin{matrix} tI_2 \\ 0 \end{matrix} & -\bar{z} & 0 \\
0 & 0 & 0 & \bar{\mu} & \tau & 0 & 0 & -\bar{v} & 0 \\
\otimes & 0 & 0 & \tau & 0 & 0 & 0 & 0 & 0 \\
\otimes & \tau I_1 & 0 & 0 & 0 & 0 & 0 & 0 & \\
\otimes & 0 & \tau I_1 & 0 & 0 & 0 & 0 & 0 & 0 \\
\bar{x}_x & \bar{w} & \bar{z} & \bar{v} & 0 & 0 & 0 & 0 & 0 \\
\otimes & \otimes & \otimes & \otimes & \otimes & \otimes & \otimes & \otimes & I_* \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I_* \mid \otimes \\
0 & 0 & 0 & 0 & 0 & I_{J1} \mid 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & I_{J2} 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0
\end{array} \qquad (4.4)$$

(rows from A to w_1)

$$D_{\text{parameters}} = \begin{array}{ccccccc}
D_A & D_{x_0} & D_{w_0} & D_{z_0} & D_{v_0} & D_{w_1} & D_{z_1} \\
I & -2A & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -2I & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -2I & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -2I & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \tau I_1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -\tau I_2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\begin{matrix} I_* \\ 0 \end{matrix} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array} \qquad (4.5)$$

$\bar{\mu} = 2 - 2\mu, \tau = (1 - t), \bar{x} = 2(x - x_1), \bar{w} = 2(w - w_2), \bar{z} = 2(z - z_2), \bar{v} = 2(v - v_1),$

If the LICQ does not hold in (x,w,z,v), the compactification constraint has to be active. Let $(\lambda,\mu^1,\mu^2,\mu)$ be the vector unequal to 0, such that:

$$
\begin{aligned}
& t\lambda\left[\left(B + B^T\right)x + q\right] + t\sum_{j\in J_{01}} \mu_j^1 B_j + t\sum_{j\in J_{02}} \mu_j^2 I_j + \mu\,\bar{x} = 0 \\
& \lambda(1 - t) + \mu\,\bar{v} = 0 \\
& (1 - t)\sum_{j\in J_{01}} \mu_j^1 I_j + \mu\,\bar{w} = 0 \\
& (1 - t)\sum_{j\in J_{02}} \mu_j^{12} I_j + \mu\,\bar{z} = 0
\end{aligned}
\tag{4.6}
$$

Without loss of generality we assume that μ = 1. Obviously: $(x,w,z;v,\lambda_0,\lambda_1,\mu^1,\mu^2,\mu)$, $\lambda_0 = 0$ the coefficient associated with the objective function of P(t) is a generalized critical point of that problem. We will prove that it is a point of Type 4 for almost all χ. Type 4 is characterized by the conditions of Remark 2.2. From the description of the problem it follows: $1 \le 1 + p \le 2n + 2 < 3n + 1$. In addition, the Jacobian of the active constraints has rank p since only the compactification constraint is able to introduce the dependence of the gradient vectors. Therefore, condition a) is satisfied for all parameter vectors. b): We will prove that the components of μ^1 and μ^2 are all non-zero. We consider the equation system describing $(x,w,z,v,0,\bar{\lambda},\bar{\mu}^1,\bar{\mu}^2,\mu)$ as a generalized critical point and the equalities $\mu^1 = 0$, $j \in (J_1 \setminus J_+(\mu^1))$, $\mu_j^2 = 0$, $j \in \left(J_2 \setminus J_+(\mu^2)\right)$. The Jacobian of this system is:

D_z	D_v	D_w	D_z	D_λ	$D^{\mu 1}$	$D^{\mu 2}$	D_t
$\otimes$	0	0	0	$\otimes$	tB_1	τI_2	$\frac{x_1 - x}{1 - t}$
0	2	0	0	τ	0	0	$\frac{2(v_2 - v)}{1 - t}$
0	0	2I	0	0	τI_1 0	0	$\frac{2(w_2 - w)}{1 - t}$ 0
0	0	0	2I	0	0	τI_2 0	$\frac{2(z_2 - z)}{1 - t}$ 0
$\otimes$	τ	0	0	0	0	0	$\frac{v_1 - v}{t}$
$\otimes$	0	τI_1 0	0	0	0	0	$\frac{(w_1 - w)}{t}$ 0
$\otimes$	0	0	τI_2	0	0	0	$\frac{(z_1 - z)}{t}$ 0
$\bar{x}_x$	$\bar{v}$	$\bar{w}$	$\bar{z}$	0	0		0
0	0	0	0	0	τI_1 0		0
0	0	0	0	0	0	τI_2 0	0

(4.7)

	D_{z^1}	D_{v^2}	D_{w^2}	D_{w^1}	D_{z^1}	D_{v^1}	
	2I	0	0	0	0	0	0
	0	2	0	0	0	0	0
	0	0	2I	0	0	0	0
constraints	0	0	0	2I	0	0	0
	0	0	0	0	0	0	0
	0	0	0	0	0	0	0
	$-2\bar{x}$	$-2\bar{v}$	$-2\bar{w}$	$-2\bar{z}$	0	0	0
	0	0	0	0	0	0	0
	0	0	0	0	0	0	0

(4.8)

We denote by (δ_x, δ_v, δ_w, $\delta_z\delta_{ig}$, δ_1, δ_2, δ_c, $\delta_\mu 1$, $\delta_\mu 1$), the coefficient vectors corresponding to the rows of the Jacobian in a linear combination of the rows equal to zero. From the columns corresponding to D_{v1}, D_{w1} , D_{z1} follows that δ_{ig},= 0, $\delta_1 = 0$, $\delta_2 = 0$, $\delta_c = 0$. Hence, from D_{x1}, D_{w2} , D_{z2} , D_{v2} it follows that: $\delta_x = (x - x_1)$, $\delta_v = (v - v_2)$, $\delta_w = (w-w_2)$, $\delta_z = (z-z_2)$. From D_t we obtain:

$||x-x_1||^2 + ||w-w_2|| + ||z-z_2||^2 + ||v-v_2||^2 = 0$ in contradiction to the fact that compactification constraint is active. From Sard's parametrized theorem we obtain that $J_1 \cup J_2 = \varnothing$ for almost all χ.

Now let us prove c). According to the latter reasoning the point (0, x, v, e, z, λ, μ^1, μ^2) is a critical point of the problem:

$$\bar{P}(t): \min t$$

$$t\lambda\left[(B+B^T)x+q\right]+t\sum_{j\in J_{01}}\mu_j^1 B_j + t\sum_{j\in J_{01}}\mu_j^2 I_j + \mu\bar{x} = 0$$

$$\lambda(1-t)+\mu\bar{v} = 0$$

$$(1-t)\sum_{j\in J_{01}}\mu_j^1 I_j + \mu\bar{w} = 0$$

$$(1-t)\sum_{j\in J_{02}}\mu_j^2 B_j + \mu\bar{z} = 0$$

$$t\left(x^T Bx + q^T x\right) + (1-t)(v-v_1) = 0$$

$$tx_i + (1-t)(z_i - (z_1)_i) = 0$$

$$t\left(B_1 x + q\right) + (1-t)(w-w_1) = 0$$

$$\|x-x_1\|^2 + (v-v_2)^2 + \|w-w_2\|^2 + \|z-z_2\|^2 = p$$

In order to see that it is non-degenerate, we will follow the same arguments as in Gómez, W. **et al**. (2000). We have to prove that for almost every χ the gradient of the objective function ($2A(x-x_0)$, $2(w-w_0)$, $2(z-z_0)$, $2(v-v_0)$) is not in the subspace generated by

$t[(B+B^T)x+q]$	tB_1	tI_2	$-2(x-x_1)$
0	$(1-t)I_1$	0	$-2(w-w_2)$
0	0	$(1-t)I_1$	$-2(z-z_2)$
$(1-t)$	0	0	$-2(v-v_2)$

Evidently, the gradient vector of the objective function belongs to the subspace generated by the gradient vector of the active constraints iff (x_0, v_0, w_0, z_0) belongs to the translated subspace. This fact implies that W (Hessian of the Lagrangian of $P_\chi(t)$ on the orthogonal subspace of the active constraints of $P_\chi(t)$ is nonsingular.

Now let us define the map

$$\Phi(\chi) = \begin{pmatrix} (x - x_0)^T A(x - x_0) + (v - v_0)^2 + \|w - w_0\|^2 + \|z - z_0\|^2 \\ t(x^T Bx + q^T x) + (1-t)(v - v_1) \\ tx + (1-t)(z - z_1) \\ t(Bx + q\ ) + (1-t)(w - w_1) \\ \|x - x_1\|^2 + (v - v_2)^2 + \|w - w_2\|^2 + \|z - z_2\|^2 - p \end{pmatrix} \tag{4.9}$$

We will consider the Euclidean norm in $\Re^{\frac{n(n+1)}{2}+8n+3}$ and the Withney topology on $[C^3_S]^{3+2n}$.

Theorem 4.2 [Genericity Theorem] Let (B,q) be fixed, then the set

$$T=\{\chi \mid P_\chi(t) \text{ is JJT-regular with respect to } (0,1)\}$$

is an open and dense set of $\Re^{\frac{n(n+1)}{2}+8n+3}$ with the topology induced by the Euclidean norm.

Proof:

T is dense: if not, there is a ball B such that $\Phi(\chi) \notin F$ for all $\chi \in B$. But B has positive Lebesgue measure, and this contradicts Theorem 4.1.

T is open : T is the preimage by Φ of F, which is an open set of $[C^3_S]^{3+2n}$ with the strong topology. Now let us prove that Φ is continuous: we consider a sequence $\chi_n \to \chi$, then it is clear that $\Phi(\chi_n)$ converges uniformly to $\Phi(\chi)$ on $\{(x,v,w,z) \ ||x - x_1||^2 + (v - v_2)^2 + ||w - w^2||^2 + ||z - z_2||^2 \leq p$. But since this set is compact it is equivalent to the convergence in the sense of the strong topology. The theorem is proved. □

5. AN ILLUSTRATIVE EXAMPLE

We consider the LCP defined by

$$B = \begin{pmatrix} -4 & 1 & 1 \\ 2 & 4 & 1 \\ 0 & 1 & 4 \end{pmatrix}, \qquad q = \begin{pmatrix} 1 \\ -6 \\ -4 \end{pmatrix}$$

B is a nondefinite matrix.

We have chosen $A = I_n$ and

$$x_0 = x_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad v_0 = 0, \quad w_0 = w_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad w_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad z_0 = z_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad z_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \cdot p = 130$$

The Figures 5.1, 5.2 and 5.3, respectively, show the curve of x_1, x_2 and x_3 corresponding to stationary points. Note that points of Type 1, 2 and 3 appear for $t \in [0,1]$.

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