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Una solución de la ecuación no estacionaria de la conducción del calor empleando Elementos Finitos con FreeFeem++.
A solution to the unsteady heat equation using Finite Element Method with FreeFeem++.

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Abstract In this work the unsteady heat equation in $2D$ is solved. Initially the time derivative was discretized using a finite difference scheme. Then the well-posed condition is proved using the variational formulation. For each time step the approximate solution is obtained by means of the Finite Element Method. The numerical computation is done by an implementation in FreeFem++.

Keywords
Unsteady Heat Equation, Finite Difference Method, Finite Element Theory, FreeFem++ software

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1. Introduction

The heat conduction problem describes the temperature distribution in a given domain over a time interval, using the thermal conductivity of the material and by a given heat source. The temperature fields are used for instance for the evaluation of the thermal stresses which have an important role in the mechanical behavior of many structural systems, particularly, for those subjected to severe non-isothermal environments.

The heat equation model can be also used by modeling diffusion processes in image analysis, in porous media, in financial mathematics amoung others. In this work we are dealing with the solution of the unsteady heat equation, starting by introducing the physical problem, after that the finite difference backward Euler discretization scheme is used to discretize the time derivative. In Section 2 we find the variational formulation for the semidiscretized equation. The well-posed condition is also proved in this section. In Section 3 we present the finite element approximation problem applying the Galerkin approach. In the last section we use the software FreeFem++ to compute the solution to this problem. In the Appendix A the required notations and mathematical resources are established.

1.1 Description of the problem

Let $\Omega \subset \mathbb{R}^2$ be a rectangle $\Omega = [0, L_x] \times [0, L_y]$. Its boundary is a closed polygonal curve Γ, divided into two parts (Γ_D, Γ_N), which cover the whole Γ and do not overlap.

We consider the initial-boundary value problem (Problem $P1^*$):

$$\frac{\partial u}{\partial t} - \nabla.(k\nabla u(t,x,y)) = f(x,y), \text{ in } \Omega \times (0,T) \tag{1}$$

$$u = g_1, \text{ in } \Gamma_D \tag{2}$$

$$\partial_n u = g_2, \text{ in } \Gamma_N \tag{3}$$

$$u(0,x,y) = u^0(x,y), \text{ in } \Omega \tag{4}$$

where u represents the temperature field, ∇ denotes de gradient operator, k is the thermal conductivity, f is the heat source, (2) is a boundary condition of Dirichlet type and (3) is a boundary condition of Neumann type, ∂_n denotes the exterior normal derivative, that is $\partial_n u = \nabla u \cdot n$ where n is the unit normal vector on points of Γ pointing always outwards. In the literature (1) is the well- known unsteady heat equation In a homogeneous medium k does not depend on the position in space. In our problem we take $k = 1$ for simplicity. Then equation (1) looks like

$$\frac{\partial u}{\partial t} - \Delta u(t,x,y) = f(x,y), \text{ in } \Omega \times (0,T), \tag{5}$$

where Δ denotes de Laplace operator, $\Delta := \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$. It is assumed that $f \in L^2(\Omega)$, $g_1 \in H^{\frac{1}{2}}(\Gamma_D)$, $g \in L^2(\Gamma_N)$. Functional spaces are defined in appendix. This equation will be integrated in time until a steady (equilibrium) solution is reached. Our task is to solve equation (5) with condtions (2)-(4).

1.2 Time discretization with backward Euler scheme

To obtain the variational formulation, we first eliminate the temporal dependence. We choose the backward Euler finite difference scheme to approximate the time derivative in a discrete time point $(n+1)$. We proceed as follows: first we take a partition in time,

$$0 = t_0 < t_1 < t_2 < \cdots < t_n < \cdots$$

where the time-step is $\delta_n = t_{n+1} - t_n$. We denote by u_n the aproximation of $u(t_n, x, y)$. Discretizing the derivative respect to the time leads us to the following approximation:

$$\frac{\partial u(t_{n+1},x,y)}{\partial t} \approx \frac{u_{n+1}-u_n}{t_{n+1}-t_n} = \frac{u_{n+1}-u_n}{\delta_n}.$$

Correspondingly, we look at the heat equation in time t_{n+1} and impose the backward Euler approximation:

$$\frac{u_{n+1}-u_n}{\delta_n} - \Delta u_{n+1} = f_{n+1}. \qquad (6)$$

with the corresponding boundary conditions

$$u_{n+1} = g_{1,n+1}, \text{ in } \Gamma_D, \qquad (7)$$
$$\partial_n u_{n+1} = g_{2,n+1}, \text{ in } \Gamma_N \qquad (8)$$

where

- $f_n = f(\cdot, t_n) : \Omega \to \mathbb{R}$
- $g_{1,n} = g_1(\cdot, t_n) : \Gamma_D \to \mathbb{R}$
- $g_{2,n} = g_2(\cdot, t_n) : \Gamma_N \to \mathbb{R}$
- $u(\cdot, t_n) \approx u_n : \Omega \to \mathbb{R}$

It means that for each time step we have to solve following equation

$$u_{n+1} - \delta_n \Delta u_{n+1} = \delta_n f_{n+1} + u_n. \qquad (9)$$

In order to apply the Finite Element Method (FEM) the variational formulation for (6) has to be found, what will be done in the next section.

2. Variational formulation

In this section the first thing is to get the variational formulation for (6) and the second one is to demostrate that this formulation is well posed. Multiplying the differential equation (6) by an arbitrary test function $v \in V_0$ (the space V_0 will be spicified below) and integrating over Ω, we find

$$\int_\Omega \frac{u_{n+1}-u_n}{\delta_n} v\, d\Omega - \int_\Omega \Delta u_{n+1} v\, d\Omega = \int_\Omega f_{n+1} v\, d\Omega. \qquad (10)$$

Consider Green's Theorem applied to the second component on the left side of (10) and multiplying by δ_n, we obtain:

$$\int_\Omega (u_{n+1}-u_n) v\, d\Omega + \delta_n \int_\Omega \nabla u_{n+1} \cdot \nabla v\, d\Omega \qquad (11)$$
$$= \int_\Omega \delta_n f_{n+1} v\, d\Omega + \delta_n \int_\Gamma (\partial_n u_{n+1}) v\, d\Gamma, \qquad (12)$$

$$\int_\Omega u_{n+1} v\, d\Omega + \delta_n \int_\Omega \nabla u_{n+1} \cdot \nabla v\, d\Omega \qquad (13)$$
$$= \int_\Omega u_n v\, d\Omega + \int_\Omega \delta_n f_{n+1} v\, d\Omega + \delta_n \int_\Gamma (\partial_n u_{n+1}) v\, d\Gamma, \qquad (14)$$

$$\int_\Omega u_{n+1} v\, d\Omega + \delta_n \int_\Omega \nabla u_{n+1} \cdot \nabla v\, d\Omega \qquad (15)$$
$$= \int_\Omega u_n v\, d\Omega + \int_\Omega \delta_n f_{n+1} v\, d\Omega \qquad (16)$$
$$+\delta_n \int_{\Gamma_D} (\partial_n u_{n+1}) v\, d\Gamma_D + \delta_n \int_{\Gamma_N} (\partial_n u_{n+1}) v\, d\Gamma_N. \qquad (17)$$

We have expressed the integral over Γ as the sum of the integrals over the two sub-boundaries, with the Dirichlet and the Neumann boundary conditions. Substituting the boundary condition, $\partial_n u_{n+1} = g_{2,n+1}$ on Γ_N.

$$\int_\Omega u_{n+1} v\, d\Omega + \delta_n \int_\Omega \nabla u_{n+1} \cdot \nabla v\, d\Omega = \int_\Omega u_n v\, d\Omega +$$
$$\int_\Omega \delta_n f_{n+1} v\, d\Omega + \delta_n \int_{\Gamma_D} (\partial_n u_{n+1}) v\, d\Gamma_D + \delta_n \int_{\Gamma_N} g_{2,n+1} v\, d\Gamma_N$$

If we impose $v = 0$ on Γ_D, then

$$\int_\Omega u_{n+1} v\, d\Omega + \delta_n \int_\Omega \nabla u_{n+1} \cdot \nabla v\, d\Omega = \int_\Omega u_n v\, d\Omega + \qquad (18)$$
$$\delta_n \int_\Omega f_{n+1} v\, d\Omega + \delta_n \int_{\Gamma_N} g_{2,n+1} v\, d\Gamma_N$$

We use (18) to convert original problem ($P1$) to weak formulation ($P2$).

Find $u_{n+1} \in H^1(\Omega)$ such that

- $u_{n+1} = g_{1,n+1}$ on Γ_D
- The equation (18) is true for $v \in V_0 = H^1_{\Gamma_D}(\Omega)$

where $f \in L^2(\Omega)$, $g_1 \in H^{\frac{1}{2}}(\Gamma_D)$, $g \in L^2(\Gamma_N)$, (see the appendix).

Note how the two boundary conditions appear in very different places of this formulation:

- The Dirichlet condition is imposed apart from the formulation and involves imposing it homogeneously to the test function v. It is called an essential boundary condition.
- The Neumann condition appears inside the formulation. It is called a natural boundary condition.

We have two spaces $V = H^1(\Omega)$, $V_0 = H^1_{\Gamma_D}$; a bilinear form $a(.,.)$ given by

$$a(u,v) = \int_\Omega u_{n+1} v\, d\Omega + \delta_n \int_\Omega \nabla u_{n+1} \cdot \nabla v\, d\Omega,$$

and a linear $l(.)$

$$l(v) = \int_\Omega u_n v\, d\Omega + \delta_n \int_\Omega f_{n+1} v\, d\Omega + \delta_n \int_{\Gamma_N} g_{2,n+1} v\, d\Gamma_N.$$

The problem admits then this simple form ($P3$). Find $u_{n+1} \in V$ such that

- $u_{n+1} = g_{1,n+1}$ on Γ_D
- $a(u_{n+1}, v) = l(v)$, $\forall v \in V_0$

By the Trace Theorem 1 (see appendix), exists $R_0 g \in H^1(\Omega)$ such that $\gamma_0(R_0 g) = g$. If we define $\widetilde{u} = u_{n+1} - R_0 g$ and substitute in $a(\cdot,\cdot)$, then we obtain:

$$\begin{aligned} a(u,v) &= \int_\Omega (\widetilde{u} + R_0 g)\, v\, d\Omega + \delta_n \int_\Omega (\nabla\widetilde{u} + \nabla R_0 g)\cdot\nabla v\, d\Omega \quad (19)\\ &= \int_\Omega \widetilde{u} v\, d\Omega + \delta_n \int_\Omega \nabla\widetilde{u}\cdot\nabla v\, d\Omega \quad (20)\\ &+ \int_\Omega R_0 g v\, d\Omega + \delta_n \int_\Omega \nabla R_0 g\cdot\nabla v\, d\Omega \quad (21)\\ &= a(\widetilde{u}, v) + a(R_0 g, v) \quad (22)\end{aligned}$$

From problem $P3$ we obtain the following problem denoted by $P4$.

Find $\widetilde{u} \in V_0$ such that,

- $a(\widetilde{u}, v) = l(v) - a(R_0 g, v)$, $\forall v \in V_0$

Building a lifting $R_0 g$ of a boundary function with an arbitrary shape can turn out to be problematic. Such task is simpler in the context of a numerical approximation, where one generally builds a lifting of an approximation of the g function. In the practice we don't construct a lifting function, we use FreeFem++ because it compute the solution without defining the lifting function. In fact, it consider the Robin condition

$$\nabla u\cdot n + \frac{1}{\varepsilon}u = \frac{1}{\varepsilon}g$$

with a very small value for ε. When ε is very small, the corresponding solution is very close to the solution with $u = g$. In this case, when dealing with a Robin condition, the functional space is $H^1(\Omega)$ and not $H^1_0(\Omega)$. Using this trick, Freefem++ is even able to impose a Dirichlet condition which does not belong to $H^{\frac{1}{2}}(\partial\Omega)$, i.e. a Dirichlet condition $g \in L^2(\partial\Omega)$ wich does not have a lift function $R_0 g \in H^1(\Omega)$.

We only use the lifting function for theoretical results.

2.1 Well-posed demonstration

In order to apply the Lax-Milgram theorem to problem $P4$, we need:

1. to prove that $a(\cdot,\cdot)$ is a bilinear continuous form:

   $$|a(u,v)| = \left|\int_\Omega uv\, d\Omega + \delta_n \int_\Omega \nabla u\cdot\nabla v\, d\Omega\right|,$$

   due to the triangular inequality we have that

   $$|a(u,v)| \le \left|\int_\Omega uv\, d\Omega\right| + \left|\delta_n \int_\Omega \nabla u\cdot\nabla v\, d\Omega\right|.$$

   Applying the Cauchy-Schwarz inequality in the term of the we obtain

   $$|a(u,v)| \le ||u||_{L^2(\Omega)}||v||_{L^2(\Omega)} + \delta_n ||\nabla u||_{L^2(\Omega)}||\nabla v||_{L^2(\Omega)}.$$

   Then

   $$\begin{aligned}|a(u,v)| &\le ||u||_{H^1}||v||_{H^1} + \delta_n ||u||_{H^1}||v||_{H^1}\\ &\le (1+\delta_n)\,||u||_{H^1}||v||_{H^1},\end{aligned}$$

   where $(1+\delta_n) \ge 0$, because $\delta_n = \Delta t > 0$.

2. We need to prove that $a(\cdot,\cdot)$ is coercive.

   $$\begin{aligned} a(u,u) &= \int_\Omega uu\, d\Omega + \delta_n \int_\Omega \nabla u\cdot\nabla u\, d\Omega\\ a(u,u) &= ||u||^2_{L^2} + \delta_n ||\nabla u||^2_{L^2},\\ a(u,u) &\ge C\left(||u||^2_{L^2} + ||\nabla u||^2_{L^2}\right),\ \ C = \text{mín}(1,\delta_n)\\ a(u,u) &\ge C||u||^2_{H^1}\end{aligned}$$

3. to prove that $l(\cdot) - a(R_0 g,\cdot)$ is a linear continuous form.

   In point 1 we demonstrate that $a(\cdot,\cdot)$ is continuous, then we only need to prove $l(\cdot)$ is a continuous form:

   $$|l(v)| = \left|\int_\Omega u_n v\, d\Omega + \delta_n \int_\Omega f_{n+1} v\, d\Omega + \delta_n \int_{\Gamma_N} g_{2,n+1} v\, d\Gamma_N\right|$$

   $$\le \left|\int_\Omega u_n v\, d\Omega\right| + \delta_n \left|\int_\Omega f_{n+1} v\, d\Omega\right| + \delta_n \left|\int_{\Gamma_N} g_{2,n+1} v\, d\Gamma_N\right|$$

   due to the triangular inequality.

   Similarly to point 1 of this section we have

   $$|l(v)| \le \underbrace{\left(||u_n||_{L^2(\Omega)}||v||_{L^2(\Omega)} + \delta_n ||f_{n+1}||_{L^2(\Omega)}||v||_{L^2(\Omega)}\right)}_{*} + \underbrace{\left(\delta_n ||g_{2,n+1}||_{L^2(\Gamma_N)}||v||_{L^2(\Gamma_N)}\right)}_{\#}$$

   We have that

   $$* \le (||u_n||_{H^1}||v||_{H^1} + \delta_n ||f_{n+1}||_{L^2}||v||_{H^1})$$

   and by the Trace Theorem

   $$\# \le \left(\delta_n\, ||g_{2,n+1}||_{L^2(\Gamma_N)}\, C||v||_{H^1}\right).$$

   Then

   $$\begin{aligned}|l(v)| &\le ||u_n||_{H^1}||v||_{H^1} + \delta_n ||f_{n+1}||_{L^2}||v||_{H^1} +\\ &\quad C\,\delta_n ||g_{2,n+1}||_{L^2(\Gamma_N)}\, ||v||_{H^1}\\ &\le \left(||u_n||_{H^1} + \delta_n ||f_{n+1}||_{L^2} + C\,\delta_n ||g_{2,n+1}||_{L^2(\Gamma_N)}\right)||v||_{H^1}\end{aligned}$$

   where $\left(||u_n||_{H^1} + \delta_n ||f_{n+1}||_{L^2} + C\,\delta_n ||g_{2,n+1}||_{L^2(\Gamma_N)}\right) \ge 0$, because $\delta_n \ge 0$ and $C \ge 0$.

Then appliyng the Lax-Milgram theorem we can say that problem $P4$ admits a unique solution.

3. Finite Element Approximation

In order to applied the Galerkin method to the given variational formulation:

Find $\widetilde{u} \in V_0$ such that

$$a(\widetilde{u}, v) = l(v) - a(R_0 g, v),\ \forall v \in V_0 \tag{23}$$

a finite dimension spaces V_h^0 has to be chosen, such that,

$$V_h^0 \subset V$$

and we obtain the following discrete problem:

$$\begin{cases} \text{Find } u_h \in V_h^0 \text{ such that} \\ a(u_h, v_h) = l(v_h),\ \forall v_h \in V_h^0 \end{cases}$$

In our case we have $V = H^1_{\Gamma_D}$ and use the $\mathbb{P}1$ finite element method.

To find the solution u_h we define the spaces

$$V_h = \{u_h \in V, u_{|K_i} \in \mathbb{P}1\}$$

and

$$V_h^{\Gamma_D} = \{v_h \in V_h | v_h = 0, \text{ on } \Gamma_D\}.$$

In the case of $\mathbb{P}1$ finite elements, the nodes a_i are the points x_i which define the mesh.

If we denote ϕ_i the shape function associated to the node x_i. Thus the functions ϕ_i are defined as

$$\phi_i(x_i) = \delta_{ij},\ \forall i, j \in \{0, \ldots, N+1\}$$

With these functions an element of V_h can be written as

$$u_h = \sum_{j\in \text{Ind}} u_j \phi_j + \sum_{j\in \text{Dir}} u_j \phi_j,\ \ u_j = u_h(x_j)$$

and an element of $V_h^{\Gamma_D}$ is of the form

$$v_h = \sum_{j\in \text{Ind}} v_j \phi_j,$$

where we will employ two lists, Dir and Ind, to number separately Dirichlet and non-Dirichlet (independent/free) nodes.

Then we substitute the discrete Dirichlet condition in the expression of u_h

$$u_h = \sum_{j\in \text{Ind}} u_j \phi_j + \sum_{j\in \text{Dir}} g_0(x_j)\phi_j.$$

Finally, if we substitute this expression in the discrete variational equation

$$\int_\Omega \nabla u_h \cdot \nabla \phi_i + c\int_\Omega u_h \phi_i = \int_\Omega f\phi_i + \int_{\Gamma_N} g_1 \phi_i,$$

applying linearity,

$$\nabla u_h = \sum_{j\in \text{Ind}} u_j \nabla\phi_j + \sum_{j\in \text{Dir}} g_0(x_j)\nabla\phi_j$$

and move to the right-hand side what we already know (the Dirichlet data), we obtain a linear system with as many equations as unknows, see [1].

The systen can be easily written in terms of the stiffness matrix

$$W_{ij} = \int_\Omega \nabla\phi_j \cdot \nabla\phi_i$$

and the mass matrix

$$M_{ij} = \int_\Omega \phi_j \cdot \phi_i,$$

using the vector

$$b_i = \int_\Omega f\phi_i + \int_{\Gamma_N} g_1 \phi_i,\ \ i \in \text{Ind},$$

to obtain

$$\sum_{j\in \text{Ind}} (W_{ij} + cM_{ij})\, u_j = b_i - \sum_{j\in \text{Dir}} (W_{ij} + cM_{ij})\, g_0(x_j),\ \ i \in \text{Ind}.$$

4. FreeFem++ Implementation

To solve numerically the unsteady heat equation, after discretizing in time, we have to do the following for each time step::

1. To mesh the domain (define the border + mesh).
2. To write the variational formulation (VF).
3. To show the result.

1. FreemFem++ define border commands, we take `A`, `B`, `C` for Neumann conditions and `D` for Dirichlet condition:

```
border A(t = 0,1){x = t; y = 0; label = 1;};
border B(t = 0,1){x = 1; y = t; label = 2;};
border C(t = 0,1){x = 1 - t; y = 1; label = 3;};
border D(t = 0,1){x = 0; y = 1 - t; label = 4;};
```

FreeFem++ mesh commands:

```
meshTh = buildmesh(A(nx) + B(ny) + C(nx)+
        D(ny));
```

2. To write the VF we need:

- To define the finite element space. For this we use
  `fespaceVh(Th,P1);`
- To define the problem to be solve

```
solve heatstep(uh,vh)
int2d(Th)(uh * vh/dt + dx(uh) * dx(vh)+
dy(uh) * dy(vh))-
int2d(Th)(u0 * vh/dt + f * vh)+
boundary conditions;
```

where we have to solve iteratively this discrete equation.

Boundary conditions are:

- Dirichlet condition
  $u = g$: `+on(BorderName,u = g)`
- Neumann condition
  $\partial_n u = g$: `-int1d(Th)(g * w)`

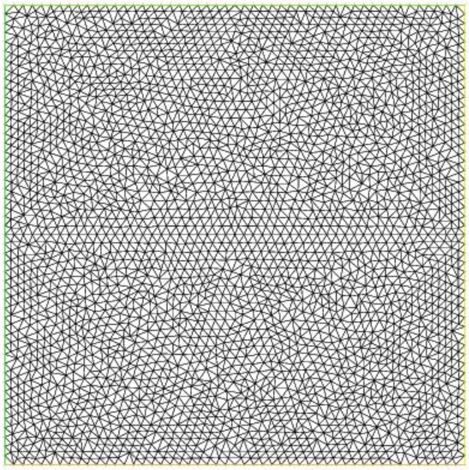

Figura 1. FreeFem++ mesh

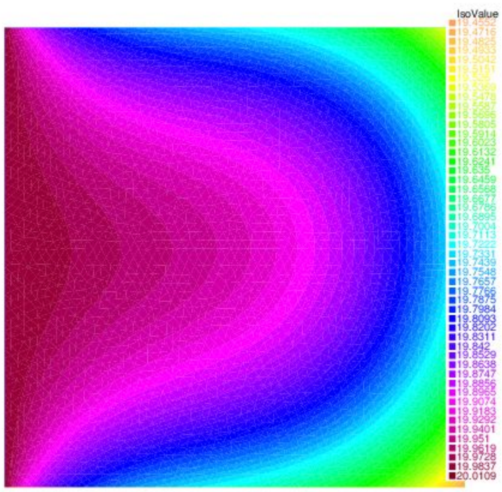

Figura 2. A time iteration solution with FreeFem++

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Mathematics definitions and results

- The space $H^{\frac{1}{2}}(\Omega)$ is define below in the Trace Theorem, (Theorem 1).

- The space $L^2(\Omega)$ is define as:

$$L^2(\Omega) = \left\{ f : \Omega \to \mathbb{R} \text{ medible}/ \int_\Omega |f|^2\, d\Omega < +\infty \right\},$$

with the norm

$$||f||_{L^2} = \left(\int_\Omega |f|^2\, d\Omega \right)^{1/2}, \forall f \in L^2(\Omega)$$

Green's first identity

$$\int_\Omega (\Delta u) v\, d\Omega + \int_\Omega \nabla u \cdot \nabla v\, d\Omega = \int_\Gamma (\partial_n u) v\, d\Gamma$$

Theorem 1 ***(Trace)***,
Let Ω be an Lipschitz-continuous bounded open subset of $\mathbb{R}^d$. The mapping

$$\gamma_0 : C^\infty\left(\overline{\Omega}\right) \to L^2\left(\partial\Omega\right)$$

$$u \hookrightarrow \gamma_0(u) = u|_{\partial\Omega}$$

is continuously extended to the Sobolev space $H^1(\Omega)$. This operator, still denoted γ_0, is called the trace operator. Besides,

- *the trace operator $\gamma_0 : H^1(\Omega) \to L^2(\partial(\Omega))$ is not surjective onto $L^2(\partial(\Omega))$;*

- *the image of $H^1(\Omega)$ by the trace operator is a fractional Sobolev space called $H^{\frac{1}{2}}(\Omega)$ which is an Hilbert space if endowed with the norm*

$$||v||_{H^{\frac{1}{2}}(\Omega)} = \inf_{u \in H^1(\Omega), \gamma_0(u) = v} ||u||_{H^1(\Omega)}$$

As a consequence, there exists a bounded (or continuous) linear operator $R_0 : H^{\frac{1}{2}}(\Omega) \to H^1(\Omega)$, called lift operator, which satisfies

$$\gamma_0 \circ R_0 = Id_{\partial\Omega}$$

Theorem 2 ***(Lax-Milgram)***, *[6]*
Let be H an Hilbert space, $a(\cdot,\cdot)$ a bilinear form on H and L a linear form on H. We assume that:

1. *$a(\cdot,\cdot)$ is continuos:*
   $\forall u, v \in H,\ |a(u,v)| \leq ||a||\,||u||_H\,||v||_H$
2. *$a(\cdot,\cdot)$ is coercive*
   $\exists \alpha > 0, \forall u \in H, a(u,v) \geq \alpha ||u||_H^2$
3. *L is continuous:*
   $\forall u \in H,\ |L(u)| \leq ||L||\,||u||_H$

We consider the abstract problem

$$(P)\begin{cases}\text{Find } u \in H \text{ such that}\\ a(u,v) = L(v), \forall v \in H\end{cases}$$

Problem (P) admits a unique solution. Moreover,

$$||u||_H \leq \frac{||L||}{\alpha}$$

If a is symmetric, then u is the unique minimizer on H of the functional

$$J(v) = \frac{1}{2}a(u,v) - L(v)$$