

SETTING OPTIMAL BOUNDS ON RISK IN ASSET ALLOCATION – A CONVEX PROGRAM

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ABSTRACT

The 'Portfolio Selection Problem' is traditionally viewed as selecting a mix of investment opportunities that maximizes the expected return subject to a bound on risk. However, in reality, portfolios are made up of a few 'asset classes' that consist of similar opportunities. The asset classes are managed by individual 'sub-managers', under guidelines set by an overall portfolio manager. Once a benchmark (the 'strategic' allocation) has been set, an overall manager may choose to allow the sub-managers some latitude in which opportunities make up the classes. He may choose some overall bound on risk (as measured by the variance) and wish to set bounds that constrain the submanagers. Mathematically we show that the problem is equivalent to finding a hyper-rectangle of maximal volume within an ellipsoid. It is a convex program, albeit with potentially a large number of constraints. We suggest a cutting plane algorithm to solve the problem and include computational results on a set of randomly generated problems as well as a real-world problem taken from the literature.

Key words: portfolio risk, strategy asset allocation, cutting plane algorithm, convex program.

MSC: 91B28.

RESUMEN

El problema de la selección del portafolio es visto tradicionalmente como hacer una selección combinada de oportunidades de inversiones las que maximizan el retorno esperado sujeto a cotas del riesgo. Sin embargo, en realidad, los portafolios están formados por unas pocas 'clases de acciones' consistentes en oportunidades similares. Las clases de acciones son manejadas por 'sub-gerentes' individuales bajo un conjunto de instrucciones establecidas por un gerente global del portafolio. Una vez un 'inicio' (afijación estratégica) ha sido fijado, un gerente global puede seleccionar el permitir a los sub-gerentes alguna libertad sobre cuales clases de oportunidades componer. Él puede seleccionar alguna cota global del riesgo (como la medida por la varianza) y desear fijar clases de cotas que restrinjan a los sub-gerentes. Matemáticamente nosotros mostramos que el problema es equivalente a hallar un hiper-rectángulo de volumen máximo dentro de un elipsoide. Este es un programa convexo dotado con potencialidades y con un gran numero de restricciones. Nosotros sugerimos un algoritmo de planos cortantes para resolver el problema e incluimos resultados computacionales sobre un gran conjunto de problemas generados aleatoriamente así como un problema del mundo real tomado de la literatura.

INTRODUCTION

The "Portfolio Selection Model" formulated by Markowitz (1952) involves the maximization of expected returns of a collection of equities (a portfolio) subject to a bound on the variance. The underlying assumption is that there is a single 'portfolio manager' who is in charge of setting the investment levels of all assets. Such an allocation will be termed a 'strategic allocation'.

In practice however, portfolios are composed of a collection of 'sub portfolios' where each sub portfolio is composed of similar assets and controlled by (possibly external) a manager with specific expertise in this class. The entire portfolio is controlled by a single overall manager who sets limits and guidelines on the sub managers but otherwise does not determine the exact mix of the sub portfolios. For example, the huge TIAA-CREF fund is composed of 4 asset classes as described in Anonymous (2001).

In this paper we assume that the entire portfolio has been set at some 'benchmark' (perhaps by some variant of the Markowitz model) and the overall manager wishes to allow the sub-managers to vary from their benchmarks (take advantage of 'tactical opportunities') while still insuring that the variance is within some given bound. Such an allocation off the benchmark (the strategic allocation) will be termed a 'tactical allocation'.

Mathematically we show that the model is equivalent to the geometric problem of finding the largest (as measured by volume) hyper-rectangle inscribed within an ellipsoid.

We will show that this is a convex program with potentially a large number of constraints. A 'cutting plane' method for its solution is proposed. This is initiated by choosing a single constraint and solving the problem subject only to this constraint. We propose a method to judiciously chose this initial constraint.

In the final section we present some computational results based on randomly generated problems.

2. THE MODEL

Let $\bar{x} \in R^n$ be the strategic allocation for the overall portfolio, where \bar{x}_i is the fraction of the portfolio invested in asset class i , for $i = 1, \dots, n$ and $e^T \bar{x} = 1$. Further assume that a portfolio manager is allowed to deviate from the strategic mix within certain limits $x^l \leq x \leq x^u$. An active allocation x over all asset classes is an allocation that is within the active bounds $x^l \leq x \leq x^u$ at any given time and the constraint $e^T \bar{x} = 1$ is relaxed. In practice this means that if $e^T \bar{x} < 1$, the portfolio is under-invested with respect to its benchmark and if $e^T \bar{x} > 1$ the portfolio is over-invested. Under-investment of the portfolio results in the remainder of the funds being kept in cash instruments. Over-investment can be financed through borrowing cash from the market or other types of financing.

Note that once the benchmark is established for a portfolio, it becomes the risk reference point. Since the performance of the portfolio is reported as the difference between the actual portfolio returns and its benchmark return (i.e. excess return), any deviation from the benchmark produces volatility of excess returns and therefore, creates risk.

There are a number of measures for portfolio risk. In this paper we define portfolio's risk as the volatility of its return as measured by its standard deviation. If $x = (x_1, \dots, x_n)$ is a portfolio invested in n different classes and Σ is their covariance matrix, then the portfolio's risk is defined as its standard deviation $\sqrt{x^T \Sigma x}$. Due to the properties of financial returns, if variances and covariances are estimated over the same time period for *all* asset classes considered in the analysis, then the corresponding matrix Σ will be positive definite (see, e.g., Korn and Korn (1968)).

For a given number σ_{SAA} (SAA stands for 'strategic asset allocation'), the set

$$S_{SAA} = \{x : x^T \Sigma x \leq \sigma_{SAA}^2\}$$

is an ellipsoid in R^n when Σ is positive definite and is symmetric about the origin in the sense that if $x \in S_{SAA}$ the so is $-x$. Likewise the set

$$S_A(\bar{x}) = \{x : (x - \bar{x})^T \Sigma (x - \bar{x}) \leq \sigma_A^2\}$$

is an ellipsoid symmetric about the point \bar{x} in the sense that if $(x - \bar{x}) \in S_{A(\bar{x})}$ the so is $-(x - \bar{x})$.

Let

$$R = \{x : x^l \leq x \leq x^u\}$$

denote a hyper-rectangle in R^n with sides perpendicular to the coordinate axes defined by the vertices x^l and x^u . These two points will represent upper and lower bounds set by the overall portfolio manager on deviations off of the benchmark \bar{x} . We are interested in finding an "optimal" pair of vectors (x^l, x^u) where the definition of optimality will now be given. In order to keep the risk within the value σ_A^2 , we will require $R \subset S_A(\bar{x})$.

There are several optimality measures that could be used here. Since we are concerned in optimizing our opportunities for the *entire* portfolio, we will seek to find the 'largest' rectangle R such that $R \subset S_A(\bar{x})$. where

'largest' is measured by volume. Since the volume of R is $\prod_{j=1}^n (x_j^u - x_j^l)$, our problem becomes

$$\max \prod_{j=1}^n (x_j^u - x_j^l)$$

$$\text{Problem P}^1: \text{ s.t. } R \subset S_A(\bar{x}) \\ u \leq v$$

Translating this last problem to center the ellipsoid $S_A(\bar{x})$ to the origin, the point x^l becomes the point $u = x^l - \bar{x}$, the point x^u becomes the point $v = x^u - \bar{x}$ and the problem becomes

$$\max \prod_{j=1}^n (v_j - u_j)$$

$$\text{Problem P}^2: \text{ s.t. } R \subset S \\ u \leq v$$

where now $R = \{x: u \leq x \leq v\}$ (this is a change of notation) and $S = S_A(0)$. Also below we will use the symbol σ instead of σ_A .

3. PROPERTIES OF THE OPTIMIZATION PROBLEM

The hyperplane $R = \{x: u \leq x \leq v\}$ has 2^n vertices (including u and v). Its center is

$$w = \frac{u+v}{2}.$$

Let $P = \text{diag}\{\pm 1\}$ denote an n by n diagonal matrix whose diagonal entries are either $+1$ or -1 . There are 2^n such matrices, and each vertex z of R has the representation

$$z = w + P \left(\frac{u-v}{2} \right) \text{ or} \\ z = \left(\frac{v+u}{2} \right) + P \left(\frac{v-u}{2} \right).$$

If the vertex $r = w + P \left(\frac{v-u}{2} \right)$ for some matrix P , then the point $s = w - P \left(\frac{v-u}{2} \right)$ is also a vertex and is opposite to r in the sense that r and s are endpoints of an interval which is a diagonal of R and passes through the center w .

In Problem 2 above, note that the constraint $R \subset S$ is satisfied if and only if all vertices of R are in S , i.e., $z^T \Sigma z \leq \sigma^2$ for all vertices z of R . This translates into the condition that

$$\left[\left(\frac{v+u}{2} \right) + P \left(\frac{v-u}{2} \right) \right]^T \Sigma \left[\left(\frac{v+u}{2} \right) + P \left(\frac{v-u}{2} \right) \right] \leq \sigma^2$$

for all diagonal ± 1 matrices P . Using the notation

$$w^+ = \left(\frac{v+u}{2} \right), \quad w^- = \left(\frac{v-u}{2} \right)$$

this condition can be written more succinctly as

$$[w^+ + Pw^-]^T \Sigma [w^+ + Pw^-] \leq \sigma^2$$

and our problem can be expressed as:

$$\max \prod_{j=1}^n w_j^-$$

$$\text{Problem } P^3: [w^+ + Pw^-]^T \Sigma [w^+ + Pw^-] \leq \sigma^2 \text{ for all } P.$$

$$w^- \geq 0$$

Using the Weierstrass Theorem (see, e.g., Bazarra and Shetty (1993), page 41), the first result that can be proven is:

Theorem 3.1: *Problem P^3 has an optimal solution for every positive real number σ , and the optimal value is positive.*

An argument using the symmetry and convexity of S implies the next result:

Theorem 3.2: *There is a hyper-rectangle of maximum volume in S that is symmetric about the origin of R^n .*

Actually the solution of our problem is unique, but this fact is not needed here.

Finally, with some (slight) abuse of the notation, our problem is equivalent to the problem:

$$\max \prod_{j=1}^n u_j$$

$$\text{Problem } P^4: (Pu)^T \Sigma (Pu) \leq \sigma^2 \text{ for all } P = \text{diag}\{\pm 1\}.$$

$$u \geq 0$$

This is equivalent to a convex problem (the log of the objective function is convex).

Using the symmetry of this problem, together with the Karush-Kuhn-Tucker conditions, we can show:

Theorem 3.3: *Let \hat{u} solve this problem for $\sigma = \hat{\sigma}$. Then for any $\tau > 0$, the solution of this problem for $\check{\sigma} = \tau \hat{\sigma}$ is $\check{u} = \tau \hat{u}$.*

The implication here is that if we solve problem P^4 for one value of σ , we have solved it for all values of σ .

The main apparent difficulty in solving this problem is that the number of constraints is potentially huge (2^n). If we knew which constraint(s) were binding, the problem would be trivial. In general, one could use a cutting plane procedure (see, e.g. Blankenship and Falk (1976)). This involves:

- For $n = 0$, select a constraint P , set $S(0) = \{P\}$, and solve the problem with only this constraint applied. Let $u_{S(0)}$ denote the optimal solution.
- Given a solution $u_{S(n)}$ with constraints in the set $S(n)$ imposed, check all other constraints for feasibility. If feasible, you are done. Otherwise choose that constraint which is most violated, add it to the set $S(n)$ to form the set $S(n + 1)$ and continue.

Unfortunately, the 'checking problem' here is a non-convex problem! As such, one would generally need to implement some type of implicit enumeration scheme such as the Falk/Soland (1969) algorithm.

Fortunately, the number n of asset classes is small in practice. Indeed the huge TIAA-CREF annuity accounts are made up of but 4 asset classes (see, e.g., Anonymous (2001)). We did solve problems of up to 15 classes (a huge number for asset classes) using complete enumeration with no numerical difficulties. The maximum number of constraints that needed to be added was 6, and the average number was 3.

In order to start the cutting plane approach, we would like to have a reasonable prediction of a good 'starting constraint'. Recall that we are trying to find the enclosed hyper-rectangle of maximum volume. If instead, we solve the problem

$$\begin{aligned} \min \quad & u^T u \\ \text{s.t.} \quad & u^T \Sigma u = \sigma^2 \end{aligned}$$

we get the largest hyper-sphere (instead of the largest hyper-rectangle) within the feasible region. It is easy to show that the optimal solution u^* is an *eigenvector* of the matrix Σ and corresponds to the largest *eigenvalue*. It turns out that this equals $1/\lambda^*$ where λ^* is the optimal KKT multiplier of the problem. The ± 1 diagonal matrix P_1 whose signs agree with u^* is used to initial the cutting plane procedure.

4. COMPUTATIONAL RESULTS

A Numerical Example with Two Asset Classes:

In this example a fund manager is performing a strategic asset allocation (SAA) between two asset classes: global fixed income and the US equity market. The fund's analyst uses the following expected returns, variances and covariances for these two asset classes:

Table 1. Risk/Return Parameters for US Equity and Fixed Income.

	Expected Return	Covariance	
US Equity	9 %	240.25	65.62
Fixed Income	6.5 %	65.62	80

The next step involves a decision on the measure of risk of the portfolio. We assume that this fund's risk is measured by the volatility (standard deviation) of its portfolio's returns.

With the above data and using Markowitz's model, the mathematical program which determines the SAA of the portfolio is

$$\begin{aligned} \max \quad & 9x_1 + 6.5x_2 \\ \text{s.t.} \quad & (x_1, x_2)\Sigma(x_1, x_2)^T \leq \sigma_S^2 \\ & x_1 + x_2 = 1 \\ & x_1, x_2 \geq 0 \end{aligned}$$

where σ_S is the desired level of the *strategic* portfolio's risk and Σ is the covariance matrix of Table 1.

Table 2 shows various solutions for the above problem for different levels of σ_S .

Assume that the portfolio's strategist allows an extra 5 % in the volatility of the portfolio return for *active* deviations σ_A . We then need to find the maximum allowable tactical deviations within this specified risk, i.e., we need to find the largest rectangle contained in the ellipse defined by the new sigma, centered at the strategic asset allocation point.

The mathematical model describing the problem is:

$$\begin{aligned} \max \quad & (x_1 - x_1^0)(x_2 - x_2^0) \\ \text{st} \quad & P(x_1 - x_1^0, x_2 - x_2^0)\Sigma(x_1 - x_1^0)(x_2 - x_2^0)P^T \leq \sigma_A^2 \\ & \text{for all } P = \text{diag}(\pm 1) \end{aligned}$$

where (x_1^0, x_2^0) is the benchmark.

Substituting $y_1 = x_1 - x_1^0$, $y_2 = x_2 - x_2^0$, the problem then becomes:

$$\begin{aligned} \max \quad & y_1 y_2 \\ \text{st} \quad & P(y_1, y_2)\Sigma(y_1, y_2)P^T \leq \sigma_A^2 \\ & \text{for all } P = \text{diag}(\pm 1) \end{aligned}$$

Table 2 presents the results of the above problem for various levels of strategic risk.

We have filled out the table by solving the problem for each level of $\sigma_A = 3, 4, 5$. (We could have used Theorem 3.3 instead).

Table 2. Strategic Asset Allocation for US Equity and Fixed Income.

Portfolio's Risk (σ_S)	Fixed Income	US Equity
10	58.75 %	41.25 %
12	33.58 %	66.42 %
14	13.63 %	86.37 %

Figure 1 illustrates the solutions graphically for $\sigma_S = 10$ and $\sigma_A = 5$.

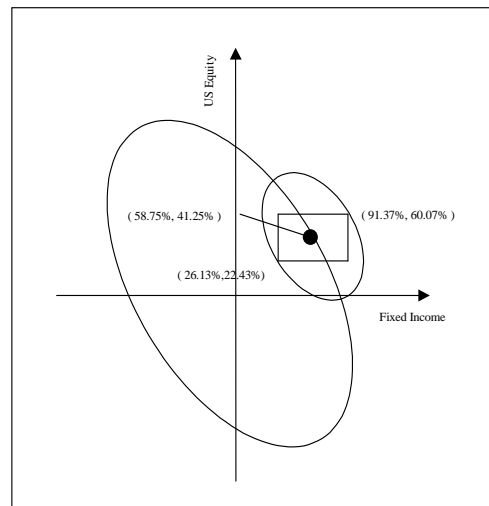


Figure 1. Optimal Deviations for US Equity and Fixed Income.

Randomly Generated Problems:

We applied our algorithm to a range of randomly generated problems with positive definite matrices and with a number of variables (asset classes) ranging from 5 to 15. Table 3 summarizes the average number of constraints added to the optimization problem, the standard deviation of that number, its maximum and running times for both optimization problems (finding a candidate solution and testing it for feasibility). The algorithm was implemented in MATLAB 5.3 and was run on a Pentium 166 MH PC. Statistical information is compiled for 25 runs for each number of variables (N).

Table 3. Algorithm Summary – Randomly Generated Problems

	Number	of	Constraints	Running	Times
No. Var.	Average	STD	Max #	Opt. Solver	Opt. Check
N = 5	1.8	0.9	3	3	0
N = 10	2.4	1.4	5	9	0.5
N = 15	2.8	1.4	6	19	50

The algorithm converged to the optimal solution on average between 2 and 3 iterations, and with 95 % of the problems all added constraints were binding. Note that the original number of constraints would have been 32, 1024 and 32768 for $N = 5, 10$ and 15 respectively. The algorithm found the optimal solution in one iteration in 44 %, 36 % and 24 % of the problems respectively.

The solution of a real-world problem can be found in Gratcheva's doctoral dissertation (2000).

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