ON THE DISTANCE FROM A POINT TO A QUADRIC SURFACE*

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ABSTRACT
In this paper we present a new algorithm to compute the Euclidean distance from a point to a quadric surface. In some sense, this algorithm is a generalization of a previous work of V. Hernández, J. Estrada and P. Barrera, where an effective algorithm to compute the Euclidean distance from a point to a plane conic is developed. It provides good approximations of the Euclidean distance from a point to a conic, as well as of the coordinates of its orthogonal projection (footpoint), even when the point is not close to the conic. In fact, the current algorithm uses the previous one and shares with it the desirable features of working well if the point may not be assumed to be very close to the quadric surface and permitting to improve iteratively the approximations, up to obtain a prescribed accuracy.

Key words: Euclidean distance, quadric surface, footpoint.

RESUMEN
En este artículo presentamos un nuevo algoritmo para calcular la distancia de un punto a una cuádrica. En cierto sentido, este algoritmo es una generalización de otro propuesto anteriormente por V. Hernandez J. Estrada y P. Barrera donde se desarrolla un algoritmo robusto para calcular la distancia euclidiana de un punto a una cónica en el plano, el cual nos permite obtener una buena aproximación de la distancia euclidiana así como las coordenadas de su proyección ortogonal (footpoint). De hecho, nuestro algoritmo utiliza el anterior en cada paso y, como este, funciona bien incluso cuando el punto exterior se encuentra alejado de la curva. También permite mejorar iterativamente la aproximación obtenida hasta lograr la exactitud que se desee.

MSC: 14Q05, 14Q10, 14Q20.

1. INTRODUCTION

There are several practical problems arising from computer graphics, computer vision, 3D robot planning pattern recognition and computational mechanics, where it is necessary to compute the Euclidean distance from a point in the three-dimensional space to an arbitrary quadric surface as well as the coordinates of the footpoint (of the orthogonal projection on the quadric). Additionally, we may not assume that the point is close to the quadric surface.

However, the above mentioned problem does not seems to be deeply studied. There are general optimization methods, which may be used to solve this problem, but it is not easy to ensure their global convergence and they may be computationally expensive. There exists methods to solve the two-dimensional problem which can be generalized to our case, but they only work when the point is close to the quadric and also avoid the computation of the coordinates of the footpoint.

In Hernández **et al.** (2002), it is presented an algorithm to compute the Euclidean distance from a point on the plane to an arbitrary conic. In that paper, it is proposed to find a suitable arc of conic limited by two points P_1 and P_2, such that the footpoint is contained in that arc. Once these points are determined they compute an initial approximation to Newton's Method using the Bisection Method on the arc. The algorithm we present here represents, in some sense, a generalization of the previous one. Moreover, we use it in each step taking advantage of its footpoint coordinates computation and accuracy. This new algorithm provides a good approximation for the Euclidean distance, even when the point is far from the given quadric. Furthermore, the approximation may be improved iteratively to attain a prescribed accuracy without increasing too much the computational cost.

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1.1 Mathematical statement of the problem

Let be Q a quadric with implicit equation:

$$f(x,y,z) = a_{200}x^2 + a_{020}y^2 + a_{002}z^2 + a_{110}xy + a_{101}xz + a_{011}yz + a_{100}x + a_{010}y + a_{001}z + a_{000} = 0 \qquad (1)$$

and

$$q = (x_0, y_0, z_0)$$

a point in the space not on Q.

By definition, the Euclidean distance from q to Q,d(q,Q), is given by:

$$d(q,Q) = \min\{\|x - p\| : f(p) = 0\} \qquad (2)$$

Thus, in order to compute the Euclidean distance we have to solve a constrained non-linear minimization problem. More geometrically, the Euclidean distance from q to Q is attained at a point p on Q such that the normal of Q at p passes through q.

If we have the quadric represented by its implicit equation (1), then the coordinates (x,y,z) of the footpoint p and the Euclidean distance d may be computed as the solution of the following nonlinear system of polynomial equations (Hoffman (1990)).

$$\begin{cases} (x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2 - d^2 = 0 \\ f(x,y,z) = 0 \\ \dfrac{\partial f}{\partial y}(p)(x-x_0) - \dfrac{\partial f}{\partial x}(p)(y-y_0) = 0 \\ \dfrac{\partial f}{\partial z}(p)(y-y_0) - \dfrac{\partial f}{\partial y}(p)(z-z_0) = 0 \end{cases} \qquad (3)$$

In the general case, for each point $q = (x_0, y_0, z_0)$ not on Q we have more than one solution of this problem. We are interested in that solution which gives us the global minimum (2). Figure 1 shows geometrically the orthogonal projection p over Q of a point q (not on Q).

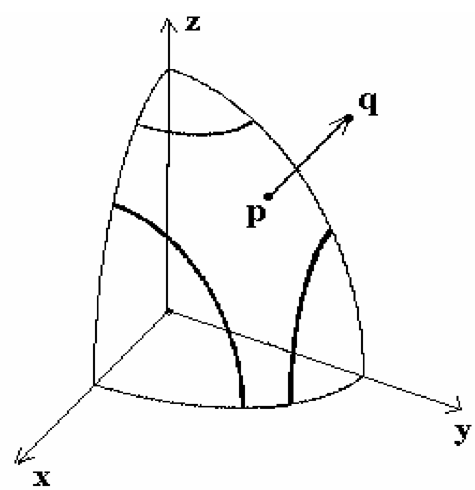

Figure 1. The orthogonal projection of q.

1.2 Other approaches

So far we know, another methods to compute the Euclidean distance from a point to a quadric are not reported in the literature. However, there are some other approaches to the distance from a point to a plane conic that can be easily generalized to solve our problem. Using elimination theory (Abhyankar (1990), Walker (1978)) we can eliminate in the system (3) the variables x, y and z and obtain a single polynomial on d whose minimal positive root d*, is the Euclidean distance from q to Q. This method generalizes that one given in Kriegman (1990) and Ponce **et al** (1992), hence it does not provide us with the footpoint coordinates computation and also may be expensive and numerically unstable (Hernández **et al**. (2002)).

In order to eliminate the last two problems, different approximations to the Euclidean distance have been considered. The simplest is the algebraic distance given by:

$$d_a(q,Q) = |f(x_0, y_0, z_0)| \qquad (4)$$

The computation of the algebraic distance is very cheap since it is given by a closed expression, but it is a poor approximation of the Euclidean distance and does not give the coordinates of the footpoint. In Taubin (1994), are introduced several approximations to the Euclidean distance from a point to an implicit curve f(x,y) = 0, if f(x,y) has continuous partial derivatives in a neighborhood of q. Generalizing this ideas to surfaces, Taubin's approximate distance of first order, δ_1 is given by:

$$\delta_1 = \frac{|F_0|}{\|F_1\|} = \frac{|f(x_0, y_0, z_0)|}{\|\nabla f(x_0, y_0, z_0)\|} \qquad (5)$$

Again, the distance is given by a closed expression and it is more precise than the algebraic distance, but q must be in a neighborhood of Q to attain a good approximation. The algorithm we propose here solves both problems, computing a very precise Euclidean distance and locating the footpoint coordinates.

2. COMPUTING THE EUCLIDEAN DISTANCE

2.1 Theoretical result

In order to compute the Euclidean distance, we have to solve the system of nonlinear equations (3). If we can compute the coordinates of the footpoint, the distance d may be obtained from the first equation of the system, so we can avoid this equation and solve the new system:

$$\begin{cases} f(x,y,z) = 0 \\ \frac{\partial f}{\partial y}(p)(x - x_0) - \frac{\partial f}{\partial x}(p)(y - y_0) = 0 \\ \frac{\partial f}{\partial z}(p)(y - y_0) - \frac{\partial f}{\partial y}(p)(z - z_0) = 0 \end{cases} \qquad (6)$$

Newton's Method may efficiently solve this system if:

i. The jacobian matrix is nonsingular in a neighborhood of the solution.

ii. A good initial approximation of the footpoint is known.

The following result is concerned with the first point. The rest of the paper will be devoted to the second one.

Theorem 1. Let be Q a quadric surface and $q = (x_0, y_0, z_0)$ a point not on Q. Then there exists a set of points on E^3 (with Lebesgue measure zero) $Z(Q)$, such that:

1. If $q \notin Z(Q)$ then the jacobian matrix of the system (6) is nonsingular in any of its solutions.

2. If $q \in Z(Q)$ then the jacobian matrix of the system (6) may be singular in some of its solutions.

Proof: Consider the system (6). The determinant of the jacobian matrix of this system J(x,y,z) is a polynomial of degree 2 in the variables x, y and z. The system (6) is singular in a solution p = (x,y,z), if and only if the polynomial system

$$\begin{cases} F_1 : f(x,y,z) = 0 \\ F_2 : \frac{\partial f}{\partial y}(p)(x - x_0) - \frac{\partial f}{\partial x}(p)(y - y_0) = 0 \\ F_3 : \frac{\partial f}{\partial z}(p)(y - y_0) - \frac{\partial f}{\partial y}(p)(z - z_0) = 0 \\ F_4 : J(x,y,z) = 0 \end{cases} \qquad (7)$$

has a solution. Using elimination theory, we may eliminate z from the pairs of equations (F_1,F_2), (F_1,F_3) and (F_1,F_4), obtaining G_1, G_2 and G_3. Now we can eliminate y from (G_1,G_2) and (G_1,G_3) obtaining two equations; finally we can eliminate x from these two equations and obtain a polynomial equation in the variables x_0, y_0 and z_0, $\Psi(x_0,y_0,z_0)$. The coefficients of Ψ depend on the coefficients of the quadric Q and (7) has a solution if (x_0,y_0,z_0) is a root of Ψ ((Abhyankar (1990), Walker (1978))). So, we can define:

$$Z(Q) = \{(x_0, y_0, z_0) : \Psi(x_0, y_0, z_0) = 0\} \tag{8}$$

Obviously, this set has Lebesgue measure zero and satisfies the thesis of the theorem.

□

Using the generalization of Bezout theorem for higher dimensions ((Abhyankar (1990)), we can see that any pair of the equations in the system (7) has at most a degree four space curve as interception. As a consequence, at most for a finite subset of points on the surface Z(Q), the jacobian matrix of the system (6) is singular and it is not possible to compute the Euclidean distance solving this system by the classic Newton's Method.

2.2. Locating the footpoint

As we know, a good initial approximation of the footpoint coordinates must be given to Newton's Method. In this section we present a simple procedure to find it.

The idea is to find three points on the quadric Q such that the footpoint is contained in the "triangle"[1] defined by these points on Q. Next we have to reduce the area of the triangle finding three new points in each step. In this way the vertices of the triangle will be closer to the footpoint. After few steps, it can be guaranteed that a vertex of the current triangle is near enough to the footpoint, so, we can take such vertex as initial approximation to Newton's Method in order to refine more this approach. Below we make a more detailed exposition.

2.2.1. Finding the three initial points

As we saw, in each stage of the procedure it is necessary to obtain three points that form a triangle over the quadric in which interior is contained the footpoint. The problem now is how to obtain these points in the first step.

To obtain the initial points (q_0^1, q_1^1, q_2^1), we first project the point q, not on Q, over each coordinate plane. These three projections of q build up the set of points needed to determine the first triangle. After that, we find in each of these planes a point q_i^1 over *Q* in the following way:

Suppose we have in each coordinate plane a point q_i^1 obtained as the orthogonal projection of q. Consider the conic curve C_i obtained as the intersection of Q with that coordinate plane. So, we compute the orthogonal projection of q_i on C_i and obtain the initial point q_i^1. Figure 2 shows how this process is done for q_0^1.

The procedure described above provides us with three initial points in all the cases of non-degenerated quadrics, except for the hyperboloid of two sheets. In that case, there are no interception points between one coordinate plane and the surface. So, there will be only two initial points. To avoid this problem, we select as third point the vertex of the hyperboloid that is at the same side as the external point, with respect to that coordinate plane.

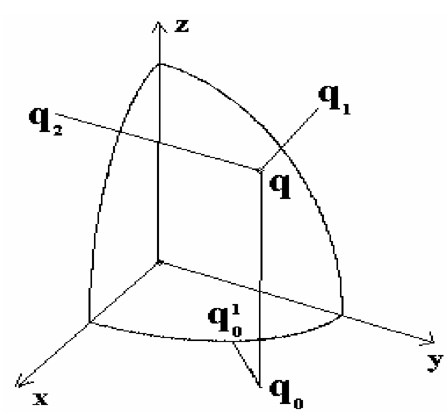

Figure 2. Selection of the first three points.

[1] By triangle we mean the part of the surface limited by three curves on Q obtained as the intersection of the surface with certain planes. These curves meet each other at a point.

2.2.2. Iteration

Assume we have three points q_0^j, q_1^j and q_2^j on the quadric Q such that the footpoint is contained in the interior of the "triangle" on Q whose boundary curves C_i^{j+1}, defined as the interception of Q with the planes $\Pi_0^{j+1}(q,q_0^j,q_1^j)$, $\Pi_1^{j+1}(q,q_1^j,q_2^j)$, and $\Pi_2^{j+1}(q,q_0^j,q_2^j)$ passing through q and every pair of these three points. On C_i^{j+1} we can find the footpoint of *q*; it will be the point q_i^{j+1} used in the next algorithm's step and so on. This process will stop when any of the new points q_i^{j+1} has the normal vector almost parallel to the line that joins it to *q*, i.e. the absolute value of the cosine of the angle between the normal of Q at q_i^{j+1} and the line $\overline{qq_i^{j+1}}$ is close to 1. Any of the vertices satisfying that condition is a good initial approximation to Newton's method. The point resulting from here is the orthogonal projection of q on Q.

3. THE ALGORITHM

In this section we resume our algorithm to compute the Euclidean distance from a point $q=(x_0,y_0,z_0)$ to a quadric Q in three-dimensional space.

Input: The vector of coefficients $\vec{a}$ of the quadric Q, a point $q=(x_0,y_0,z_0)$ and the termination criteria $N_1, N_2, \varepsilon_1, \varepsilon_2$.

Output: The Euclidean distance d from q to Q and the coordinates of the orthogonal projection p of q on Q.

1. Compute the linear change of coordinates T(x,y,z) that reduces Q to the canonical form Q'.

2. Initial approximation

(a) *Computing the first three points:*

i. Compute the orthogonal projection q_i of q on the coordinate planes $\Pi_i^0, i=1,2,3$.
ii. On each coordinate plane define the conic curve $C_i^0 = Q' \cap \Pi_i^0$.
iii. Compute the first point q_i^1 as the orthogonal projection of q_i in the conic curve C_i^0. See Hernández **et al.** (2002).

(b) *Iteration:* For j = 1 ... N_1 do

i. Compute the equation of the planes Π_1^j defined by the points (q,q_1^j,q_2^j), Π_2^j defined by (q,q_2^j,q_3^j) and Π_3^j defined by (q,q_1^j,q_3^j).
ii. On each plane Π_i^j define the conic curve $C_i^j = Q' \cap \Pi_i^j$.
iii. Compute the point q_i^{j+1} as the orthogonal projection of the external point q in the conic curve C_i^j. C_i^0. See Hernández **et al.** (2002).
iv. Compute the normal vector to Q' at q_i^j, $n_i^j = \dfrac{\nabla F(q_i^j)}{\|\nabla F(q_i^j)\|}$.
v. Compute the cosine of the angle θ_i^j between the vector parallel to the line joining q and q_i^j and the normal vector to Q' at q_i^j, n_i^j:
$$\cos\theta_i^j = \frac{q-q_i^j}{\|q-q_i^j\|} n_i^j \tag{9}$$
vi. If for any j, $\theta_i^j > 1-\varepsilon_1$, then put $p' = q_i^j$, **END**. Else, set j = j+1 and return to i.
vii. If N_1+1 then select p' as the point $q_i^{N_1}$ corresponding to $\max\{|\cos\theta_i^{N_1}|, i=1,2,3\}$ (10) and set $p' = q_i^{N_1}$.

3. **Newton's Method** (To solve (6) for the quadric Q', from the initial approximation $p_0 = p'$).

For j = 0...N_2 do

(a) Compute Δp^j as the solution of the linear system $J(p^j)\Delta p^j = F(p^j)$ where J is the jacobian matrix of (6) and $(F_1,F_2,F_3)^T$ is the vector of the left side of (6).

(b) Correct the position of P^j, $P^{j+1} = p^j + \Delta p^j$.

(c) Obtain the relative error $e^j = \dfrac{\| \Delta p^j \|}{\| p^{j+1} \|}$.

(d) If $e^j < \varepsilon_2$ then **END,** else set j = j+1 and return to (a).

4. Set $p = T^{-1}(p')$.

5. Compute $d(q,Q) = \| q - p \|$.

4. CONVERGENCE

It was presented in section 3 an algorithm to compute the algebraic distance from a point to a quadric surface. It basically consists of two iterative processes: one to find an initial approximation to Newton's Method and then Newton's Method itself, which converges if a good initial approximation is given. So, we just have to prove the convergence of the first one.

4.1 Global convergence Theorem

In order to proof the convergence, we need to recall first some classical definitions and theorems related to optimization theory.

Definition 1. An algorithm A is a mapping defined on a space X that assigns to every point $x \in X$ a subset of X.

Definition 2. A point-to-set mapping A from X to Y is said to be closed at $x \in X$ if the assumptions

i. $x_k \to x$, $x_k \in X$

ii. $y_k \to y$, $y_k \in A(x_k)$

imply

iii. . $y \in A(x)$

Note that a continuous point-to-point mapping is always closed. The converse is, however, not true in general (see Luenberger (1973)).

Theorem 2 (Global convergence theorem) Let A be an algorithm on X, and suppose that, given x_0, the sequence $\{x_k\}_{k=0}^{\infty}$ is generated satisfying $x_{k+1} \in A(x_k)$. Let a solution set $\Gamma \subset X$ be given, and suppose:

i. All points x_k are contained in a compact set $S \subset X$.

ii. There is a continuous real function Z on X such that

  1. If $x \notin \Gamma$, then $Z(y) < Z(x)$ for all $y \in A(x)$.

  2. If $x \in \Gamma$, then $Z(y) \le Z(x)$ for all $y \in A(x)$.

iii. The mapping A is closed at points outside Γ.

Then the limit of any convergent subsequence of x_k is a solution.

A proof of this theorem can be found in Luenberger (1973).

4.2. The Algorithm Converges

To prove the convergence of our algorithm, we only need to check the hypothesis of the global convergence theorem. Consider the set $X = Q \times Q \times Q$, now we can write our algorithm in the language of the global convergence theorem as:

$$A : X \to \wp(X),$$

$$A(p_0, p_1, p_2) = \{(m(p_0, p_1), m(p_1, p_2), m(p_0, p_2))\},$$

where $\wp(X)$ is the power set of X and $m(p_i, p_j)$ is the point where the minimum distance to q over the segment of the conic $Q \cap \pi$ between p_i and p_j is reached, and p_i is the plane defined by q, p_i and p_j.

Note that A can be seen as a point-to-point map and given any $X_0 = (p_0^0, p_1^0, p_2^0)$ we have only one way of construct a sequence x_k such that $x_{k+1} \in x_k$, and it reproduces the algorithm proposed on this paper. Of course, the set Γ is {(p,p,p)} where p is the footpoint on Q; in some very special cases this set may contain more than one element, but we will choose only the first footpoint that we find, since all of them give us the same distance from q. Let's check the other hypothesis:

i. It is obvious since all the points in the sequence x_k belong to the ball of radius $\| x_0 \|_\infty$ centered at the origin in E^9 if Q is centered at the origin in E^3.
ii. Defining $Z(p_0, p_1, p_2) = \| q - p_0 \|_2 + \| q - p_1 \|_2 + \| q - p_2 \|_2$, the item holds.
iii. In order to prove that the map A is closed we can prove just that if we have the sequences $R_k \to R$, $T_k \to T$ and $S_k = m(R_k, T_k) \to S$, where $R_k, T_k \in Q$, then $S = m(R, T)$.

Obviously, *S* belongs to the plane generated by q, R and T, $\pi(q,R,T)$, so it belongs to the conic $Q \cap \pi(q, R, T)$. Suppose now that exists a point on Q, such that $H = m(R, T)$ and $H \neq S$. Consider the continuous function $d_q(x) = \| q - x \|_2$ which give us the distance from x to q and is defined over E^3, for all $\varepsilon > 0$ it can be found δ_1 and δ_2 such that:

$$\| S - x \| < \delta_1 \Rightarrow | d_q(S) - d_q(x) | < \varepsilon \tag{11}$$

and

$$\| H - x \| < \delta_2 \Rightarrow | d_q(H) - d_q(x) | < \varepsilon \tag{12}$$

Take $\delta = \min\left\{ \delta_1, \delta_2, \frac{\| S - H \|_2}{2} \right\}$ and k very large so that $\| S - S_k \| < \delta$ and $\| H - H_k \| < \delta$ for some $H_k \in Q \cap \pi(q, R_k, T_k)$. Then, we can state that:

$$| d_q(S) - d_q(S_k) | < \varepsilon \tag{13}$$

$$| d_q(H) - d_q(H_k) | < \varepsilon \tag{14}$$

Since the minimum is reached at H and S_k for $Q \cap \pi(q, R, S)$ and $Q \cap \pi(q, R_k, S_k)$ respectively, we can choose ε such that,

$$d_q(H) + \varepsilon < d_q(S) \tag{15}$$

$$d_q(S_k) + \varepsilon < d_q(H_k) \tag{16}$$

by the inequalities (13) and (14) respectively we have:

$$d_q(S_k) - \varepsilon < d_q(S) < d_q(S_k) + \varepsilon \qquad (17)$$

$$d_q(H) - \varepsilon < d_q(H_k) < d_q(H) + \varepsilon \qquad (18)$$

from (16), the second inequality of (18) and (15) we have:

$$d_q(S_k) + \varepsilon < d_q(H_k) < d_q(H) + \varepsilon < d_q(S) \qquad (19)$$

i.e.

$$d_q(S_k) + \varepsilon < d_q(S) \qquad (20)$$

Since this last equation and the second inequality of (17) are contradictory we conclude that H = S and the map A is closed.

With this result we can assure the convergence of the algorithm proposed even when Newton's Method can't be performed. In these few cases it could be applied this first approach until a good approximation is found without applying Newton's Method.

5. EXPERIMENTAL RESULTS

In this section we compare the relative errors associated with different approximations of the Euclidean distance from a point to a quadric. In the following table the relative errors resulting from this new algorithm can be compared with the relative errors resulting from the previously mentioned approximations to the Euclidean Distance for different quadrics. The column "ealg" corresponds to the relative errors of the algebraic distance, "etau1" to the relative errors of the Taubin's first order approximation, and "enew" to the relative errors of the distance computed by the new algorithm. The implicit equations of the quadrics selected for the numerical experiment are:

Ellipsoid:	$36x^2 + 25y^2 + 25z^2 = 769$
Hyperboloid of one sheet:	$9x^2 + 4y^2 + 9z^2 = 36$
Hyperboloid of two sheet:	$9x^2 + 4y^2 - 9z^2 = 36$
Elliptic paraboloid:	$x^2 + y^2 = 4z$
Hyperbolic paraboloid:	$9x^2 - 4y^2 = 36z$

Quadric Surface	External point	Euclidean distance	Relative errors		
			ealg	Etau1	enew
Ellipsoid	(156,153,204)	293.62	8516.4	0.4987	$*1.586 \times 10^{-12}$
Hyp. (1S)	(1.24,3.32,0.75)	0.4722	0.0034	0.0389	$*2.203 \times 10^{-9}$
Hyp. (2S)	(4.89,3,4)	1.5703	2.2995	0.0441	$*1.413 \times 10^{-16}$
Par. (ell)	(15,15,49)	7.1414	7.8918	0.1626	$*2.846 \times 10^{-16}$
Par. (hip)	(6.33,2.22,1.83)	2.7222	1.8086	0.1341	$*3.271 \times 10^{-16}$

As it can be seen, the best results (marked with "*") correspond to the new algorithm. It gives us very good approximations even when the point is far from the surface.

Figure 3 shows a massive use of our algorithm. For a uniform grid of 1000 points generated in the cube $[0,5]^3$ it was computed and displayed the corresponding footpoints with respect to the unitary sphere. For almost all the points, the number of iterations N_1 required to compute the initial approximation to Newton's Method was equal to 3, and in all the cases it was less than 6. All relative errors were less than $\varepsilon_2 = 10^{-8}$.

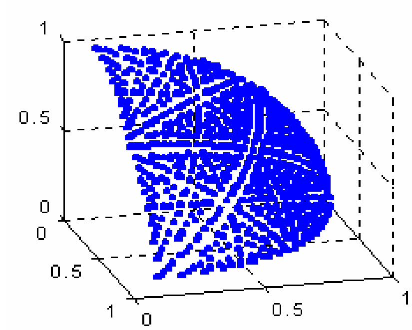

Figure 3. The footpoints of a grid.

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